

STABILITY CONDITIONS AND RELATED FILTRATIONS FOR (G, h)-CONSTELLATIONS

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ABSTRACT. Given an infinite reductive algebraic group G , we consider G -equivariant coherent sheaves with prescribed multiplicities, called (G, h) -constellations, for which two stability notions arise. The first one is analogous to the θ -stability defined for quiver representations by King [Kin94] and for G -constellations by Craw and Ishii [CI04], but depending on infinitely many parameters. The second one comes from Geometric Invariant Theory in the construction of a moduli space for (G, h) -constellations, and depends on some finite subset D of the isomorphy classes of irreducible representations of G . We show that these two stability notions do not coincide, answering negatively a question raised in [BT15]. Also, we construct Harder-Narasimhan filtrations for (G, h) -constellations with respect to both stability notions (namely, the μ_θ -HN and μ_D -HN filtrations). Even though these filtrations do not coincide in general, we prove that they are strongly related: the μ_θ -HN filtration is a sub-filtration of the μ_D -HN filtration, and the polygons of the μ_D -HN filtrations converge to the polygon of the μ_θ -HN filtration when D grows.

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INTRODUCTION

In moduli problems, we usually consider objects on which we impose a certain stability condition to be able to construct a moduli space, i.e., a space parametrizing stable or semistable objects. When using Geometric Invariant Theory (in the following GIT to abbreviate) to construct such a moduli space, another notion of stability shows up, the so-called GIT-stability, and one has to work out the relation between those two stability notions in order to apply GIT to construct the moduli space of (semi)stable objects we are interested in. In this article, we consider a new moduli problem treated in [BT15], where the objects are certain coherent sheaves for which the stability condition depends on infinitely many parameters and do not exactly match with the GIT-stability condition. We investigate in detail the two stability conditions and notice remarkable phenomena, in particular at the level of the corresponding Harder-Narasimhan filtrations. Moreover, we answer several questions which remained unresolved in [BT15].

Let G be a complex reductive algebraic group, and let X be an affine G -scheme of finite type. When G is finite, Craw and Ishii [CI04] generalized the notion of G -cluster on X . They defined a G -constellation on X as a G -equivariant coherent \mathcal{O}_X -module \mathcal{F} with global sections $H^0(\mathcal{F})$ isomorphic to the regular representation of G as a G -module. Then they defined a stability condition on G -constellations, namely the θ -stability, and they constructed the moduli space of θ -(semi)stable G -constellations on X by following ideas of King [Kin94]. The key ingredient in this construction is the reformulation of the θ -stability condition into a GIT-stability condition. Finally, they proved that minimal resolutions of singularities of certain quotients X/G can be obtained as the moduli space of θ -stable G -constellations on X for some θ .

Let us now assume that G is infinite. In [BT15], Becker and the first-named author defined similar concepts and constructed the moduli space of θ -stable (G, h) -constellations on X , where $h : \text{Irr } G \rightarrow \mathbb{N}_{\geq 0}$ is a function that assigns a non-negative integer to each irreducible G -module and replaces the regular representation in this new setting; see §1 for details. This moduli space, say $M_\theta(X)$, is a generalization of the invariant Hilbert scheme of Alexeev and Brion [AB05], and one hopes that minimal resolutions of singularities of certain categorical quotients $X//G$ are isomorphic to $M_\theta(X)$ for some θ .

The methods used in [CI04] to construct $M_\theta(X)$ no longer apply when G is infinite since the θ -stability depends now on infinitely many parameters (one for each irreducible G -module), and it is not clear whether the θ -stability condition can still be expressed as a GIT-stability condition (which depends only on a finite number of parameters). Nevertheless, a GIT-stability condition was defined in [BT15, §2], depending on a finite subset $D \subset \text{Irr } G$, and it was proved that for D big enough in $\text{Irr } G$, the θ -stability of a (G, h) -constellation implies its GIT-stability. The possible converse implication and the relations between θ -semistability and GIT-semistability were addressed in [BT15, §5] but these questions remained unanswered at that time. In this article, we will see that those two stability conditions are actually different.

Proposition A (§3.1 and §4). *The notions of θ -(semi)stability and GIT-(semi)stability mentioned above for (G, h) -constellations do not coincide.*

This proposition answers negatively [BT15, Question 5.2] and implies that the GIT approach used in [BT15] to construct $M_\theta(X)$ is unsuitable to construct the moduli space of θ -semistable (G, h) -constellations on X ; in particular, the answer to [BT15, Question 5.1] is also negative.

Once we know that those two stability conditions do not coincide, it is natural to compare them. The first step in this article is to reformulate the θ -stability and the GIT-stability in terms of slope stability conditions, giving rise to the μ_θ -stability and the μ_D -stability (where the index D is to emphasize the dependence on D). The advantage of dealing with these new stability conditions defined by slopes is that we can then construct for any (G, h) -constellation the so-called *Harder-Narasimhan filtration* [HN75]. Within the years, the latter has been proved to be an extremely useful tool in the study of properties of moduli spaces in algebraic geometry. The Harder-Narasimhan filtration is defined recursively by considering at each step the maximal destabilizing subobject; see §1.2 for a precise definition. In some sense, this filtration measures how far an object is from being semistable. Therefore, comparing stability conditions almost boils down to comparing the corresponding Harder-Narasimhan filtrations for each object. In §3.3, we will explain how to associate to each Harder-Narasimhan filtration a polygon which encodes the numerical data of the filtration. The next statement gathers our results.

Theorem B. *Let G be an infinite reductive algebraic group acting on an affine scheme of finite type X , let $h : \text{Irr } G \rightarrow \mathbb{N}_{\geq 0}$ be a Hilbert function, and let \mathcal{F} be a (G, h) -constellation on X . Let $D \subset \text{Irr } G$ be a finite subset satisfying Hypothesis 2.5. Then the following holds:*

- (i) \mathcal{F} admits a μ_θ -Harder-Narasimhan filtration \mathcal{F}_\bullet (Theorem 1.7) as well as a μ_D -Harder-Narasimhan filtration \mathcal{G}_\bullet^D (Theorem 2.17).
- (ii) If the finite subset $D \subset \text{Irr } G$ is big enough, then \mathcal{F}_\bullet is a subfiltration of \mathcal{G}_\bullet^D (Theorem 3.3). Moreover, \mathcal{G}_\bullet^D is a subfiltration of some Jordan-Hölder filtration of the μ_θ -semistable factors of \mathcal{F} (Remark 3.4).
- (iii) Even though the μ_D -Harder-Narasimhan filtration might not stabilize when $D \subset \text{Irr } G$ grows (§4), the sequence of polygons associated with $(\mathcal{G}_\bullet^D)_{D \subset \text{Irr } G}$ converges to the polygon associated with \mathcal{F}_\bullet when D grows (Theorem 3.7).

The fact that the (G, h) -constellations we consider here (those generated in D_- , see Definition 1.3) do not form an abelian category prevents us from applying the results of [Rud97] to obtain directly the existence of the Harder-Narasimhan filtrations. Actually, we have to substantially modify the classical proofs for existence and uniqueness of Harder-Narasimhan filtrations in our situation.

Let us also mention that for GIT-unstable (G, h) -constellations, it is possible to apply the results of Kempf [Kem78] to find a unique 1-parameter subgroup which is maximally destabilizing in the GIT sense. Gómez, Sols, and the second-named author constructed in [GSZ15], out of Kempf's result, the so-called *Kempf filtration* (which would depend on D in our situation) and proved that, for coherent torsion free sheaves on smooth projective varieties, this filtration coincides with the Harder-Narasimhan filtration by relating the convexity properties of both filtrations. This method can be performed in other moduli problems; see for instance [Zam14] for the case of quiver representations. However, in our situation, the Kempf filtration and the μ_D -Harder-Narasimhan filtration seem to be unrelated even though we lack explicit examples to support this claim.

The paper is organized as follows. In §1 we introduce (G, h) -constellations and θ -stability. Then we convert the θ -stability into a slope stability condition, the μ_θ -stability, and we prove the existence and uniqueness of the μ_θ -Harder-Narasimhan filtration for any (G, h) -constellation.

Then, §2 is devoted to recall the construction of the moduli space of θ -stable (G, h) -constellations as in [BT15]. When performing this construction by using GIT, a GIT-stability condition is introduced, which depends on a choice of a finite subset $D \subset \text{Irr } G$. Then, as for the θ -stability, we convert this GIT-stability condition into a slope stability condition, the μ_D -stability, and we construct the μ_D -Harder-Narasimhan filtration.

In §3, which is the heart of this article, we study closely the relations between θ -stability and GIT-stability. First, we summarize the implications between the different stability notions considered in this article and answer related questions raised in [BT15, §5]. Then, once we know that θ -stability and GIT-stability do not coincide for (G, h) -constellations, it is natural to compare the corresponding Harder-Narasimhan filtrations. We make explicit the relations between these two filtrations and prove parts (ii) and (iii) of Theorem B.

Finally, §4 illustrates Proposition A and Theorem B by providing examples of the different phenomena that can occur.

Acknowledgements. The first-named author is grateful to Christian Lehn for suggesting the correct definition of θ -stability in the setting of (G, h) -constellations and for helpful discussions. The second-named author thanks the Johannes Gutenberg Universität in Mainz for the hospitality it provided while part of this work was done. The first-named author benefits from the support of the DFG via the SFB/TR 45 "Periods, Moduli Spaces and Arithmetic of Algebraic Varieties". The second-named author is supported by the project "Comunidade Portuguesa de Geometria Algebraica" PTDC/MAT-GEO/0675/2012 funded by Portuguese FCT and project MTM2013-42135-P granted by the Spanish Ministerio de Economía y Competitividad.

Notation. Throughout this article we work over the field of complex numbers \mathbb{C} . Let G be an infinite reductive algebraic group. Then we denote by $\text{Irr } G$ the set of isomorphism classes of irreducible G -modules $\rho : G \rightarrow \text{GL}(V_\rho)$, and by $R(G) = \bigoplus_{\rho \in \text{Irr } G} \mathbb{N} \cdot \rho$ the representation monoid of G . An element of $R(G)$ identifies naturally with a function $h : \text{Irr } G \rightarrow \mathbb{N}$; we call such a function a *Hilbert function*. Let X be an affine G -scheme of finite type. We say that \mathcal{F} is an (\mathcal{O}_X, G) -module if \mathcal{F} is a G -equivariant coherent \mathcal{O}_X -module whose module of global sections $H^0(\mathcal{F})$ is a G -module with finite multiplicities. We denote the category of (\mathcal{O}_X, G) -modules by $\text{Coh}^G(X)$. We say that h is the Hilbert function of \mathcal{F} if the multiplicities of the G -module $H^0(\mathcal{F})$ are given by h .

Whenever the word *(semi)stable* appears in the text, or the abbreviation $(s)s$, two statements should be read: a first one for *stable* or s , and a second one for *semistable* or ss . If they appear together with the symbols \succeq or \preceq , one should read $>$ or $<$ with *stable*, and \geq or \leq with *semistable*.

1. CONSTELLATIONS AND μ_θ -HARDER-NARASIMHAN FILTRATION

We fix once and for all an (possibly non-connected) infinite reductive algebraic group G , an affine G -scheme of finite type X , and a non-zero Hilbert function $h : \text{Irr } G \rightarrow \mathbb{N}$.

1.1. Constellations and θ -stability. In this subsection we present the notions of (G, h) -constellation, θ -stability, and μ_θ -stability.

Definition 1.1. A (G, h) -constellation on X is an (\mathcal{O}_X, G) -module \mathcal{F} such that

$$H^0(\mathcal{F}) = \bigoplus_{\rho \in \text{Irr } G} \mathcal{F}_\rho \otimes V_\rho \cong \bigoplus_{\rho \in \text{Irr } G} V_\rho^{h(\rho)}$$

as a G -module, i.e., the multiplicities of the G -module $H^0(\mathcal{F})$ are given by the Hilbert function h .

Moduli spaces parametrizing (G, h) -constellations are constructed in [BT15]. As the set of all (G, h) -constellations on X is too large in general to be parametrized by a scheme, the moduli problem is restricted to consider (G, h) -constellations satisfying a certain stability condition, the θ -stability, that we now introduce.

Definition 1.2. Let $\theta = (\theta_\rho)_{\rho \in \text{Irr } G}$ be a sequence of rational numbers (which depends on the Hilbert function h) satisfying:

- $\theta_\rho < 0$ for only finitely many $\rho \in \text{Irr } G$;
- $\theta_\rho > 0$ for infinitely many $\rho \in \text{Irr } G$;
- If $h(\rho) = 0$, then $\theta_\rho = 0$; and
- $\sum_{\rho \in \text{Irr } G} \theta_\rho h(\rho) = 0$.

Then we call *stability function* $\theta : R(G) \rightarrow \mathbb{R} \cup \{\infty\}$ the function defined by

$$\theta(W) := \langle \theta, h_W \rangle := \sum_{\rho \in \text{Irr } G} \theta_\rho \cdot \dim W_\rho,$$

where $W = \bigoplus_{\rho \in \text{Irr } G} W_\rho \otimes V_\rho$ is the isotypic decomposition of W .

In order to consider θ as a function $\text{Coh}^G(X) \rightarrow \mathbb{R} \cup \{\infty\}$, we set

$$\theta(\mathcal{F}) := \theta(H^0(\mathcal{F})) = \sum_{\rho \in \text{Irr } G} \theta_\rho \cdot \dim \mathcal{F}_\rho.$$

In particular, if \mathcal{F} is a (G, h) -constellation, then we have

$$\theta(\mathcal{F}) = \sum_{\rho \in \text{Irr } G} \theta_\rho h(\rho) = 0.$$

The choice of θ induces a decomposition

$$(1) \quad \text{Irr } G = D_+ \sqcup D_0 \sqcup D_- \quad \text{such that} \quad \theta_\rho \begin{cases} > 0 & \text{if } \rho \in D_+ \\ = 0 & \text{if } \rho \in D_0 \\ < 0 & \text{if } \rho \in D_- \end{cases}$$

It follows from the definition of θ that D_- is finite, D_+ is infinite, and the sets $\text{supp } h \cap D_-$ and $\text{supp } h \cap D_+$ are non-empty, where $\text{supp } h := \{\rho \in \text{Irr } G \mid h(\rho) \neq 0\}$.

Definition 1.3. Let θ be as in Definition 1.2, and let \mathcal{F} be an (\mathcal{O}_X, G) -module. If \mathcal{F} is generated by its negative part $\bigoplus_{\rho \in D_-} \mathcal{F}_\rho \otimes V_\rho$ as an \mathcal{O}_X -module, then we say that \mathcal{F} is *generated in D_-* .

Even though (G, h) -constellations need not be generated in D_- in general, we will only consider in this article those generated in D_- . The reason to do that comes from the GIT construction of the moduli spaces of (G, h) -constellations in [BT15]; this will become clear with the introduction of the invariant Quot scheme in §2.1.

Warning. *From now on, (G, h) -constellations are always assumed to be generated in D_- (with respect to a given θ as in Definition 1.2). On the other hand, (\mathcal{O}_X, G) -modules are not assumed to be generated in D_- , except when we say so.*

Definition 1.4. Let \mathcal{F} be an (\mathcal{O}_X, G) -module with a non-zero negative part, but not necessarily generated in D_- . Then we define

$$r(\mathcal{F}) = \sum_{\rho \in D_-} \dim \mathcal{F}_\rho \in \mathbb{N}_{>0},$$

and the θ -slope of \mathcal{F} , where θ is a stability function as in Definition 1.2, by

$$\mu_\theta(\mathcal{F}) := \frac{-\theta(\mathcal{F})}{r(\mathcal{F})} \in \mathbb{R}.$$

We now define two stability conditions on (\mathcal{O}_X, G) -modules generated in D_- , which will turn out to be equivalent for (G, h) -constellations.

Definition 1.5. Let θ be as in Definition 1.2, and let \mathcal{F} be an (\mathcal{O}_X, G) -module generated in D_- .

- (1) \mathcal{F} is called θ -(semi)stable if $\theta(\mathcal{F}) = 0$ and for every (\mathcal{O}_X, G) -submodule $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$ generated in D_- we have

$$\theta(\mathcal{F}') \geq 0.$$

- (2) \mathcal{F} is called μ_θ -(semi)stable if for all (\mathcal{O}_X, G) -submodule $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$ generated in D_- we have

$$\mu_\theta(\mathcal{F}') \leq \mu_\theta(\mathcal{F}).$$

If \mathcal{F} is non θ -semistable resp. non μ_θ -semistable, we say that \mathcal{F} is θ -unstable resp. μ_θ -unstable.

Lemma 1.6. *Let θ be as in Definition 1.2, and let \mathcal{F} be an (\mathcal{O}_X, G) -module generated in D_- . Then \mathcal{F} is θ -(semi)stable if and only if \mathcal{F} is μ_θ -(semi)stable and $\theta(\mathcal{F}) = 0$. In particular, the notions of θ -(semi)stability and μ_θ -(semi)stability are equivalent for (G, h) -constellations.*

Proof. This follows immediately from Definition 1.5. □

The θ -stability condition is the stability condition used in [BT15] to prove the existence of a moduli space of stable (G, h) -constellations; see [BT15, Theorem 4.3]. The reason for considering μ_θ -stability instead of θ -stability –which coincide for (G, h) -constellations by Lemma 1.6– is that this reformulation in terms of slopes will allow us to talk about Harder-Narasimhan filtrations.

1.2. μ_θ -Harder-Narasimhan filtration. In this subsection, we construct the μ_θ -Harder-Narasimhan filtration for a (G, h) -constellation (Theorem 1.7). We follow the classical treatment, see for instance [HL10, §1.3], but conveniently adapted to our situation. We first show the existence of a unique maximal destabilizing sub-object (Proposition 1.10), and then we proceed by induction to prove the existence and uniqueness (which is a consequence of Proposition 1.11) of the μ_θ -Harder-Narasimhan filtration.

Theorem 1.7. *Let μ_θ be as in Definition 1.4, and let \mathcal{F} be a (G, h) -constellation. Then \mathcal{F} has a unique filtration*

$$0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_t \subsetneq \mathcal{F}_{t+1} = \mathcal{F}$$

verifying

- (i) each \mathcal{F}_i is an (\mathcal{O}_X, G) -submodule generated in D_- ;
- (ii) each quotient $\mathcal{F}^i := \mathcal{F}_i / \mathcal{F}_{i-1}$ is μ_θ -semistable; and
- (iii) the slopes of the quotients are strictly decreasing

$$\mu_\theta(\mathcal{F}^1) > \mu_\theta(\mathcal{F}^2) > \cdots > \mu_\theta(\mathcal{F}^t) > \mu_\theta(\mathcal{F}^{t+1}).$$

We call this filtration the μ_θ -Harder-Narasimhan filtration (μ_θ -HN filtration for short) of \mathcal{F} . The integer $t + 1$ is called the length of the filtration.

Let us note that the μ_θ -HN filtration of a (G, h) -constellation \mathcal{F} is trivial if and only if \mathcal{F} is μ_θ -semistable. Explicit examples of (G, h) -constellations with non trivial μ_θ -HN filtration will be computed in §4.

The proof of Theorem 1.7 is postponed till the end of the subsection. It is clear from the definition that every (\mathcal{O}_X, G) -module generated in D_- is a (G, \tilde{h}) -constellation for a certain Hilbert function \tilde{h} . We will actually prove Theorem 1.7 for an arbitrary (\mathcal{O}_X, G) -module generated in D_- even though we formulate it for (G, h) -constellations which are the objects we are interested in.

First, we prove a lemma –that we call *seesaw property* following the terminology in [Rud97]– which relates slopes of objects in exact sequences.

Lemma 1.8. (seesaw property) *Given a short exact sequence*

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

of (\mathcal{O}_X, G) -modules, all with non-zero negative part, we have

$$\mu_\theta(\mathcal{F}') \leq \mu_\theta(\mathcal{F}) \iff \mu_\theta(\mathcal{F}') \leq \mu_\theta(\mathcal{F}'') \iff \mu_\theta(\mathcal{F}) \leq \mu_\theta(\mathcal{F}'') .$$

Moreover, if any of the inequalities is an equality, the other two are also equalities.

Proof. Denote by h' , h , and h'' the Hilbert functions of \mathcal{F}' , \mathcal{F} , and \mathcal{F}'' respectively. Since G is a reductive group, it is clear that $h(\rho) = h'(\rho) + h''(\rho)$ for all $\rho \in \text{Irr } G$. Then we have $\theta(\mathcal{F}) = \theta(\mathcal{F}') + \theta(\mathcal{F}'')$ and $r(\mathcal{F}) = r(\mathcal{F}') + r(\mathcal{F}'')$. The assumption of having a non-zero negative part guarantees that $r(\mathcal{F})$, $r(\mathcal{F}')$, and $r(\mathcal{F}'')$ are non-zero. It follows that

$$\mu_\theta(\mathcal{F}) = \frac{-\theta(\mathcal{F})}{r(\mathcal{F})} = \frac{-\theta(\mathcal{F}') - \theta(\mathcal{F}'')}{r(\mathcal{F}') + r(\mathcal{F}'')},$$

whence

$$-\theta(\mathcal{F})r(\mathcal{F}') + \theta(\mathcal{F}')r(\mathcal{F}) = -\theta(\mathcal{F}'')r(\mathcal{F}) + \theta(\mathcal{F})r(\mathcal{F}'') ,$$

which implies

$$-\theta(\mathcal{F})r(\mathcal{F}') + \theta(\mathcal{F}')r(\mathcal{F}) \geq 0 \iff -\theta(\mathcal{F}'')r(\mathcal{F}) + \theta(\mathcal{F})r(\mathcal{F}'') \geq 0.$$

The last equivalence turns out to be

$$\mu_\theta(\mathcal{F}') \leq \mu_\theta(\mathcal{F}) \iff \mu_\theta(\mathcal{F}) \leq \mu_\theta(\mathcal{F}'') .$$

A similar treatment shows the equivalence of these two inequalities with the other one. Finally, we note that all implications still hold if we replace equalities by inequalities. \square

We now recall a result expressing the finiteness of the different functions which can appear as Hilbert functions of subobjects.

Proposition 1.9. ([BT15, Prop. 1.9]) *Let \tilde{h} be an arbitrary Hilbert function. There is a finite set of Hilbert functions $\{h_1, \dots, h_N\}$ such that for any (G, \tilde{h}) -constellation \mathcal{F} and any (\mathcal{O}_X, G) -submodule $\mathcal{F}' \subseteq \mathcal{F}$ generated in D_- , the Hilbert function h' of \mathcal{F}' is one of the h_1, \dots, h_N .*

Among the set of Hilbert functions, there exists a partial order defined by:

$$(2) \quad h_1 \geq h_2 \iff \forall \rho \in \text{Irr } G, \quad h_1(\rho) \geq h_2(\rho).$$

Let us note the following basic facts which will be very useful in the remaining of this subsection: If there is an inclusion of (\mathcal{O}_X, G) -modules $\mathcal{F}_1 \subseteq \mathcal{F}_2$ with Hilbert functions h_1 and h_2 respectively, then $h_1 \leq h_2$. Moreover, if $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $\mathcal{F}_1 = \mathcal{F}_2$ if and only if $h_1 = h_2$.

The next result guarantees the existence of a unique maximal μ_θ -destabilizing subobject for (\mathcal{O}_X, G) -modules generated in D_- .

Proposition 1.10. *Let \mathcal{F} be an (\mathcal{O}_X, G) -module generated in D_- . Then there exists a unique (\mathcal{O}_X, G) -submodule $\mathcal{F}' \subseteq \mathcal{F}$ generated in D_- such that:*

- (i) *if $0 \neq \mathcal{G} \subseteq \mathcal{F}$ is generated in D_- , then $\mu_\theta(\mathcal{G}) \leq \mu_\theta(\mathcal{F}')$; and*
- (ii) *if $0 \neq \mathcal{G} \subseteq \mathcal{F}$ is generated in D_- and $\mu_\theta(\mathcal{G}) = \mu_\theta(\mathcal{F}')$, then $\mathcal{G} \subseteq \mathcal{F}'$.*

Proof. It is clear that if \mathcal{F}' exists then it has to be unique by (ii). Let us prove the existence of \mathcal{F}' .

With the notation of Proposition 1.9 applied to h , let $M := \max\{\mu_\theta(h_i)\}_{i=1}^N$. Among the Hilbert functions satisfying $\mu_\theta(h_i) = M$, we pick one which is maximal for the partial order defined by (2). Let h' be such a Hilbert function, and let $\mathcal{F}' \subseteq \mathcal{F}$ be an (\mathcal{O}_X, G) -submodule generated in D_- whose Hilbert function is h' . Then by construction \mathcal{F}' satisfies (i). A priori h' and \mathcal{F}' are not unique, however we will show that \mathcal{F}' satisfies (ii), and this will imply the uniqueness of h' and \mathcal{F}' .

Let us now prove that \mathcal{F}' satisfies (ii). Let $0 \neq \mathcal{G} \subseteq \mathcal{F}$ be an (\mathcal{O}_X, G) -submodule generated in D_- such that $\mu_\theta(\mathcal{G}) = \mu_\theta(\mathcal{F}')$. Consider the exact sequence of (\mathcal{O}_X, G) -modules:

$$(3) \quad 0 \rightarrow \mathcal{F}' \cap \mathcal{G} \rightarrow \mathcal{F}' \oplus \mathcal{G} \rightarrow \mathcal{F}' + \mathcal{G} \rightarrow 0,$$

where $\mathcal{F}' \oplus \mathcal{G}$ and $\mathcal{F}' + \mathcal{G}$ are generated in D_- (since \mathcal{F}' and \mathcal{G} also are) but not necessarily $\mathcal{F}' \cap \mathcal{G}$. We denote by h_1, h_2, h_3 , and h_4 the Hilbert functions of \mathcal{F}' , \mathcal{G} , $\mathcal{F}' + \mathcal{G}$, and $\mathcal{F}' \cap \mathcal{G}$ respectively. We distinguish between two cases.

- (a) If the negative part of $\mathcal{F}' \cap \mathcal{G}$ is zero, i.e., if $h_4(\rho) = 0$ for all $\rho \in D_-$. Then we deduce from (3) that $h_1(\rho) + h_2(\rho) = h_3(\rho)$ for all $\rho \in D_-$. It follows that

- $r(\mathcal{F}' + \mathcal{G}) = r(h_3) = r(h_1 + h_2) = r(\mathcal{F}' \oplus \mathcal{G})$;
- $\sum_{\rho \in D_-} \theta_\rho h_3(\rho) = \sum_{\rho \in D_-} \theta_\rho (h_1(\rho) + h_2(\rho))$; and
- $\sum_{\rho \in D_+} \theta_\rho h_3(\rho) \leq \sum_{\rho \in D_+} \theta_\rho (h_1(\rho) + h_2(\rho))$.

Hence

$$\mu_\theta(\mathcal{F}' + \mathcal{G}) = \frac{-\theta(\mathcal{F}' + \mathcal{G})}{r(\mathcal{F}' + \mathcal{G})} \geq \frac{-\theta(\mathcal{F}' \oplus \mathcal{G})}{r(\mathcal{F}' \oplus \mathcal{G})} = \mu_\theta(\mathcal{F}' \oplus \mathcal{G}).$$

Now since $\mu_\theta(\mathcal{F}') = \mu_\theta(\mathcal{G})$, the seesaw property applied to

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F}' \oplus \mathcal{G} \rightarrow \mathcal{G} \rightarrow 0$$

gives $\mu_\theta(\mathcal{F}' \oplus \mathcal{G}) = \mu_\theta(\mathcal{F}')$. Therefore $\mathcal{F}' + \mathcal{G}$ is an (\mathcal{O}_X, G) -submodule of \mathcal{F} with greater μ_θ -slope and whose Hilbert function h_3 is greater or equal to h_1 for the partial order defined by (2). By definition of \mathcal{F}' , we must have $h_1 = h_3$; this implies that $\mathcal{F}' + \mathcal{G} = \mathcal{F}'$, i.e., that $\mathcal{G} \subseteq \mathcal{F}'$.

- (b) If the negative part of $\mathcal{F}' \cap \mathcal{G}$ is non-zero, then we denote by $\widetilde{\mathcal{F}' \cap \mathcal{G}}$ the (\mathcal{O}_X, G) -submodule generated by its negative part; $\widetilde{\mathcal{F}' \cap \mathcal{G}}$ can be $\mathcal{F}' \cap \mathcal{G}$ itself or a non-zero proper subsheaf. We denote the Hilbert function of $\widetilde{\mathcal{F}' \cap \mathcal{G}}$ by h_5 . By definition, we have $h_5(\rho) = h_4(\rho)$ for all $\rho \in D_-$, and $h_5(\rho) \leq h_4(\rho)$ for all $\rho \in \text{Irr } G \setminus D_-$. So arguing as before, we easily prove that

$$\mu_\theta(\widetilde{\mathcal{F}' \cap \mathcal{G}}) \geq \mu_\theta(\mathcal{F}' \cap \mathcal{G}).$$

Suppose that $\mathcal{G} \not\subseteq \mathcal{F}'$, and consider the two exact sequences of non-zero (\mathcal{O}_X, G) -modules:

$$(4) \quad 0 \rightarrow \mathcal{F}' \cap \mathcal{G} \rightarrow \mathcal{G} \rightarrow \mathcal{U} \rightarrow 0,$$

and

$$(5) \quad 0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F}' + \mathcal{G} \rightarrow \mathcal{U} \rightarrow 0.$$

Since $\mu_\theta(\mathcal{G}) = \mu_\theta(\mathcal{F}')$ is maximal among (\mathcal{O}_X, G) -submodules of \mathcal{F} generated in D_- , we necessarily have $\mu_\theta(\mathcal{G}) \geq \mu_\theta(\widetilde{\mathcal{F}' \cap \mathcal{G}})$. On the other hand, we just saw that $\mu_\theta(\widetilde{\mathcal{F}' \cap \mathcal{G}}) \geq \mu_\theta(\mathcal{F}' \cap \mathcal{G})$, hence $\mu_\theta(\mathcal{G}) \geq \mu_\theta(\mathcal{F}' \cap \mathcal{G})$. The seesaw property applied to (4) gives $\mu_\theta(\mathcal{G}) \leq \mu_\theta(\mathcal{U})$. Thus, since $\mu_\theta(\mathcal{F}') = \mu_\theta(\mathcal{G})$, the seesaw property applied to (5) gives $\mu_\theta(\mathcal{F}') \leq \mu_\theta(\mathcal{F}' + \mathcal{G})$. By definition of \mathcal{F}' , this implies that $\mathcal{F}' + \mathcal{G} = \mathcal{F}'$, which contradicts our assumption $\mathcal{G} \not\subseteq \mathcal{F}'$.

Therefore, if $0 \neq \mathcal{G} \subseteq \mathcal{F}$ is an (\mathcal{O}_X, G) -submodule generated in D_- such that $\mu_\theta(\mathcal{G}) = \mu_\theta(\mathcal{F}')$, then necessarily $\mathcal{G} \subseteq \mathcal{F}'$. \square

The next proposition assures the uniqueness of the first term of the μ_θ -HN filtration.

Proposition 1.11. *Let \mathcal{F} be an (\mathcal{O}_X, G) -module generated in D_- with a filtration satisfying the properties (i)-(iii) of Theorem 1.7. Then the first term \mathcal{F}_1 of the filtration is the (\mathcal{O}_X, G) -submodule \mathcal{F}' given by Proposition 1.10.*

Proof. The proof goes by induction on the length $t + 1$ of the filtration. If $t = 0$, then \mathcal{F} is μ_θ -semistable and $\mathcal{F}_1 = \mathcal{F} = \mathcal{F}'$. We now suppose that $t \geq 1$, and we consider the filtration of length t given by

$$0 \subsetneq \mathcal{F}_2/\mathcal{F}_1 \subsetneq \mathcal{F}_3/\mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_t/\mathcal{F}_1 \subsetneq \mathcal{F}_{t+1}/\mathcal{F}_1 = \mathcal{F}/\mathcal{F}_1.$$

Then it is clear that properties (i)-(iii) of Theorem 1.7 are again satisfied. Hence, by induction hypothesis, we know that $\mathcal{F}_2/\mathcal{F}_1 = (\mathcal{F}/\mathcal{F}_1)'$. We want to deduce from this that $\mathcal{F}_1 = \mathcal{F}'$, i.e., that \mathcal{F}_1 satisfies the properties (i) and (ii) of Proposition 1.10. Let $0 \neq \mathcal{G} \subseteq \mathcal{F}$ be an (\mathcal{O}_X, G) -submodule generated in D_- . We distinguish between several cases:

- a) If $\mathcal{G} \subset \mathcal{F}_1$, then $\mu_\theta(\mathcal{G}) \leq \mu_\theta(\mathcal{F}_1)$ since \mathcal{F}_1 is μ_θ -semistable.
b) If $\mathcal{G} \not\subset \mathcal{F}_1$, then $\mathcal{G}/(\mathcal{F}_1 \cap \mathcal{G})$ is a non-zero (\mathcal{O}_X, G) -submodule of $\mathcal{F}/\mathcal{F}_1$ generated in D_- , and thus

$$\mu_\theta(\mathcal{G}/(\mathcal{F}_1 \cap \mathcal{G})) \leq \mu_\theta(\mathcal{F}_2/\mathcal{F}_1) < \mu_\theta(\mathcal{F}_1),$$

where the first inequality is by induction hypothesis and the second inequality is property (iii) of Theorem 1.7. We denote by $\widetilde{\mathcal{F}_1 \cap \mathcal{G}}$ the (\mathcal{O}_X, G) -submodule generated by the negative part of $\mathcal{F}_1 \cap \mathcal{G}$.

- If $\widetilde{\mathcal{F}_1 \cap \mathcal{G}} = 0$, then an explicit calculation gives $\mu_\theta(\mathcal{G}) \leq \mu_\theta(\mathcal{G}/(\mathcal{F}_1 \cap \mathcal{G}))$, and thus $\mu_\theta(\mathcal{G}) < \mu_\theta(\mathcal{F}_1)$.
- If $\widetilde{\mathcal{F}_1 \cap \mathcal{G}} \neq 0$, then one easily checks that $\mu_\theta(\mathcal{F}_1 \cap \mathcal{G}) \leq \mu_\theta(\widetilde{\mathcal{F}_1 \cap \mathcal{G}})$. Since \mathcal{F}_1 is μ_θ -semistable, we have $\mu_\theta(\widetilde{\mathcal{F}_1 \cap \mathcal{G}}) \leq \mu_\theta(\mathcal{F}_1)$, and thus $\mu_\theta(\mathcal{F}_1 \cap \mathcal{G}) \leq \mu_\theta(\mathcal{F}_1)$. The seesaw property applied to

$$(6) \quad 0 \rightarrow \mathcal{F}_1 \cap \mathcal{G} \rightarrow \mathcal{G} \rightarrow \mathcal{G}/(\mathcal{F}_1 \cap \mathcal{G}) \rightarrow 0$$

implies that either

$$\mu_\theta(\mathcal{G}) \leq \mu_\theta(\mathcal{G}/(\mathcal{F}_1 \cap \mathcal{G})) < \mu_\theta(\mathcal{F}_1)$$

or

$$\mu_\theta(\mathcal{G}) < \mu_\theta(\mathcal{F}_1 \cap \mathcal{G}) \leq \mu_\theta(\mathcal{F}_1).$$

In all cases, we get that $\mu_\theta(\mathcal{G}) \leq \mu_\theta(\mathcal{F}_1)$, i.e., that \mathcal{F}_1 satisfies (i) of Proposition 1.10.

Let us now suppose that \mathcal{G} satisfies $\mu_\theta(\mathcal{G}) = \mu_\theta(\mathcal{F}_1)$ but is not contained in \mathcal{F}_1 . We have seen in b) that either $\widetilde{\mathcal{F}_1 \cap \mathcal{G}} = 0$, and then

$$\mu_\theta(\mathcal{G}) \leq \mu_\theta(\mathcal{G}/(\mathcal{F}_1 \cap \mathcal{G})) < \mu_\theta(\mathcal{F}_1)$$

which is a contradiction, or else $\widetilde{\mathcal{F}_1 \cap \mathcal{G}} \neq 0$, and then

$$\mu_\theta(\mathcal{F}_1 \cap \mathcal{G}) \leq \mu_\theta(\mathcal{F}_1) = \mu_\theta(\mathcal{G}).$$

In the second case, the seesaw property applied to (6) gives $\mu_\theta(\mathcal{G}) \leq \mu_\theta(\mathcal{G}/(\mathcal{F}_1 \cap \mathcal{G}))$. But $\mu_\theta(\mathcal{F}_2/\mathcal{F}_1) < \mu_\theta(\mathcal{F}_1) = \mu_\theta(\mathcal{G})$, hence $\mu_\theta(\mathcal{F}_2/\mathcal{F}_1) < \mu_\theta(\mathcal{G}/(\mathcal{F}_1 \cap \mathcal{G}))$, which contradicts our induction assumption. Therefore $\mathcal{G} \subset \mathcal{F}_1$, and thus \mathcal{F}_1 satisfies (ii) of Proposition 1.10. Then the result follows from the uniqueness of an (\mathcal{O}_X, G) -submodule of \mathcal{F} generated in D_- and satisfying properties (i) and (ii) of Proposition 1.10. \square

Proof of Theorem 1.7. Let us prove the existence of a filtration satisfying properties (i)-(iii) of Theorem 1.7 by induction on the dimension of the negative part of \mathcal{F} . If \mathcal{F} is a μ_θ -semistable (\mathcal{O}_X, G) -module, then the filtration $0 \subsetneq \mathcal{F}$ satisfies (i)-(iii). Otherwise, let \mathcal{F}' be the (\mathcal{O}_X, G) -submodule given by Proposition 1.10. Then $0 < r(\mathcal{F}/\mathcal{F}') < r(\mathcal{F})$, and thus by induction hypothesis, there exists a filtration

$$0 \subsetneq \overline{\mathcal{F}}_1 \subsetneq \overline{\mathcal{F}}_2 \subsetneq \cdots \subsetneq \overline{\mathcal{F}}_{t-1} \subsetneq \overline{\mathcal{F}}_t = \mathcal{F}/\mathcal{F}',$$

for some $t \geq 1$, verifying the assumptions (i)-(iii) of Theorem 1.7. For $i \geq 2$, we denote by \mathcal{F}_i the preimage of $\overline{\mathcal{F}}_{i-1}$ in \mathcal{F} and we denote $\mathcal{F}_1 := \mathcal{F}'$. Then one easily checks that the filtration

$$0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_t \subsetneq \mathcal{F}_{t+1} = \mathcal{F}$$

also satisfies properties (i)-(iii) of Theorem 1.7.

It remains to prove the uniqueness part of Theorem 1.7, but this is a direct consequence of Proposition 1.11. \square

Remark 1.12. Using the same arguments as for Proposition 1.10, we can prove that every μ_θ -semistable (G, h) -constellation \mathcal{F} has a (generally non-unique) μ_θ -Jordan-Hölder filtration, i.e., a filtration

$$0 \subsetneq \mathcal{J}_1 \subsetneq \mathcal{J}_2 \subsetneq \cdots \subsetneq \mathcal{J}_s \subsetneq \mathcal{J}_{s+1} = \mathcal{F}$$

verifying that all $\mathcal{J}_i/\mathcal{J}_{i-1}$ are μ_θ -stable and $\mu_\theta(\mathcal{J}_1) = \mu_\theta(\mathcal{J}_2) = \cdots = \mu_\theta(\mathcal{J}_{s+1})$. Then the graded object $\bigoplus_{i=1}^{s+1} \mathcal{J}_i/\mathcal{J}_{i-1}$ is unique, i.e., it does not depend (up to isomorphism) on the choice of the μ_θ -Jordan-Hölder filtration.

2. GIT-STABILITY AND μ_D -HARDER-NARASIMHAN FILTRATION

In this section we introduce the notions of GIT-stability and μ_D -stability for (G, h) -constellations. In §2.1 we introduce the invariant Quot scheme $\text{Quot}^G(\mathcal{H}, h)$, then in §2.2 we explain how to identify the (G, h) -constellations with certain elements of $\text{Quot}^G(\mathcal{H}, h)$. In particular, we will see that isomorphism classes of (G, h) -constellations are in one-to-one correspondence with certain Γ -orbits of $\text{Quot}^G(\mathcal{H}, h)$, where Γ is a reductive algebraic group acting on $\text{Quot}^G(\mathcal{H}, h)$ defined in §2.3. It is then natural in §§2.3–2.4 to consider the GIT-quotient of $\text{Quot}^G(\mathcal{H}, h)$ by the Γ -action, and that is how the GIT-stability comes into the picture. Indeed, the invariant Quot scheme $\text{Quot}^G(\mathcal{H}, h)$ being quasi-projective, we need to restrict the Γ -action to the open subset of GIT-(semi)stable points $\text{Quot}^G(\mathcal{H}, h)^{(s)s}$ to obtain a categorical quotient. The correspondence between elements of $\text{Quot}^G(\mathcal{H}, h)$ and (G, h) -constellations, in turn, allows us to talk about GIT-(semi)stable (G, h) -constellations. In §2.5 we introduce a new stability condition on the (\mathcal{O}_X, G) -modules generated in D_- , the μ_D -stability, which is a slope stability condition. Finally, we prove that μ_D -stability and GIT-stability coincide for (G, h) -constellations, and we construct the μ_D -Harder-Narasimhan filtration associated with a (G, h) -constellation.

Let us mention that §§2.1–2.4 are mainly extracted from [BT15, §2 and §3], but §2.5, which is the most important part of this section, is an original work.

As before, we fix a Hilbert function $h : \text{Irr } G \rightarrow \mathbb{N}$ and a stability function θ ; see Definition 1.2. The classical reference for the concepts related to Geometric Invariant Theory is [MFK94].

2.1. The invariant Quot scheme. For every $\rho \in D_-$, let $A_\rho = \mathbb{C}^{h(\rho)}$. We define the G -equivariant free \mathcal{O}_X -module of finite rank

$$(7) \quad \mathcal{H} := \left(\bigoplus_{\rho \in D_-} A_\rho \otimes V_\rho \right) \otimes \mathcal{O}_X,$$

and we denote by $\text{Quot}^G(\mathcal{H}, h)$ the *invariant Quot scheme* which parametrizes all the (\mathcal{O}_X, G) -submodules $\mathcal{K} \subseteq \mathcal{H}$ such that \mathcal{H}/\mathcal{K} is a (G, h) -constellation. Equivalently, the invariant Quot scheme parametrizes the equivalence classes of quotient maps $[q : \mathcal{H} \twoheadrightarrow \mathcal{F}]$, where \mathcal{F} is a (G, h) -constellation; two quotients q and q' being in the same equivalence class if $\text{Ker } q = \text{Ker } q'$.

The invariant Quot scheme was constructed by Jansou in [Jan06], and then used in [BT15] to construct the moduli space of θ -stable (G, h) -constellations. In

the next subsection, we will explain how to associate a given (G, h) -constellation \mathcal{F} (generated in D_- by assumption, see §1.1) with a quotient $[q : \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \text{Quot}^G(\mathcal{H}, h)$. Let us emphasize that there is not such a correspondence for (G, h) -constellations not generated in D_- , and this is the reason why we consider only (G, h) -constellations generated in D_- in this article.

2.2. Quotients originating from a constellation. Let \mathcal{F} be a (G, h) -constellation, and let $H^0(\mathcal{F}) = \bigoplus_{\rho \in \text{Irr } G} \mathcal{F}_\rho \otimes V_\rho$ be the isotypic decomposition of its space of global sections. Since $\mathcal{F}_\rho = \text{Hom}_G(V_\rho, H^0(\mathcal{F}))$, we have evaluation maps

$$(8) \quad \text{ev}_\rho : \mathcal{F}_\rho \otimes V_\rho \otimes H^0(\mathcal{O}_X) \rightarrow H^0(\mathcal{F}), \quad \alpha \otimes v \otimes f \mapsto f \cdot \alpha(v),$$

and $H^0(\mathcal{F})$ is generated as an $H^0(\mathcal{O}_X)$ -module by the images of ev_ρ ($\rho \in D_-$) by assumption. We choose a basis of each \mathcal{F}_ρ , i.e., we fix an isomorphism $\psi_\rho : A_\rho \rightarrow \mathcal{F}_\rho$, and we compose it with the evaluation map (8) considered as a map between \mathcal{O}_X -modules. We obtain

$$(9) \quad q_\rho : A_\rho \otimes V_\rho \otimes \mathcal{O}_X \rightarrow \mathcal{F}, \quad a \otimes v \otimes f \mapsto f \cdot \psi_\rho(a)(v).$$

Their sum

$$q := \bigoplus_{\rho \in D_-} q_\rho : \mathcal{H} = \bigoplus_{\rho \in D_-} A_\rho \otimes V_\rho \otimes \mathcal{O}_X \rightarrow \mathcal{F}$$

gives us a point $[q : \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \text{Quot}^G(\mathcal{H}, h)$ with the property that the map

$$(10) \quad \varphi_\rho : A_\rho \rightarrow \mathcal{F}_\rho = \text{Hom}_G(V_\rho, H^0(\mathcal{F})), \quad a \mapsto (v \mapsto q(a \otimes v \otimes 1)),$$

is just the isomorphism ψ_ρ since, for $a \in A_\rho$ and $v \in V_\rho$, we have

$$\varphi_\rho(a)(v) = q(a \otimes v \otimes 1) = 1 \cdot \psi_\rho(a)(v) = \psi_\rho(a)(v).$$

Definition 2.1. Let $[q : \mathcal{H} \twoheadrightarrow \mathcal{F}]$ be an element of the invariant Quot scheme $\text{Quot}^G(\mathcal{H}, h)$. If for every $\rho \in D_-$ the map φ_ρ defined by (10) is an isomorphism, then we say that q *originates from \mathcal{F}* .

Different choices of bases for \mathcal{F}_ρ give different elements of $\text{Quot}^G(\mathcal{H}, h)$, and it is precisely to cancel this ambiguity that we will introduce in §2.3 the action of the group Γ on $\text{Quot}^G(\mathcal{H}, h)$.

Conversely, given an element $[q : \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \text{Quot}^G(\mathcal{H}, h)$, the quotient \mathcal{F} is a (G, h) -constellation. However, the induced maps φ_ρ need not to be isomorphisms so that $[q]$ need not to originate from \mathcal{F} as above.

Lemma 2.2. *With the notation above, the subset*

$$\Omega^G(\mathcal{H}, h) := \{ [q : \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \text{Quot}^G(\mathcal{H}, h) \mid [q] \text{ originates from } \mathcal{F} \}$$

is open in the invariant Quot scheme $\text{Quot}^G(\mathcal{H}, h)$.

Proof. By the discussion above, we have

$$\Omega^G(\mathcal{H}, h) = \bigcap_{\rho \in D_-} \{ [q] \in \text{Quot}^G(\mathcal{H}, h) \mid \text{rk } \varphi_\rho = h(\rho) \},$$

where φ_ρ is the linear map defined by (10). Since $\text{rk } \varphi_\rho = h(\rho)$ is an open condition for each $\rho \in D_-$, we obtain the result. \square

Remark 2.3. As $\text{Quot}^G(\mathcal{H}, h)$ is reducible in general, $\Omega^G(\mathcal{H}, h)$ might not be a dense open subset.

2.3. GIT setting. Consider the natural action of the group $\Gamma' := \prod_{\rho \in D_-} \mathrm{GL}(A_\rho)$ on \mathcal{H} by multiplication from the left on the constituent components. This action induces an action on $\mathrm{Quot}^G(\mathcal{H}, h)$ from the right, which we describe. Let $\gamma = (\gamma_\rho)_{\rho \in D_-} \in \Gamma'$ and $[q: \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H}, h)$. Then $[q] \cdot \gamma$ is the map

$$[q] \cdot \gamma: \mathcal{H} \twoheadrightarrow \mathcal{F}, \quad a_\rho \otimes v_\rho \otimes f \mapsto q(\gamma_\rho a_\rho \otimes v_\rho \otimes f).$$

As the subgroup of scalar matrices $K := \{\prod_{\rho \in D_-} \alpha \mathrm{Id}_{A_\rho}; \alpha \in \mathbb{C}^*\} \cong \mathbb{C}^*$ acts trivially on $\mathrm{Quot}^G(\mathcal{H}, h)$, we restrict to consider the action of the subgroup

$$(11) \quad \Gamma := \left\{ (\gamma_\rho)_{\rho \in D_-} \in \prod_{\rho \in D_-} \mathrm{GL}(A_\rho) \left| \prod_{\rho \in D_-} \det(\gamma_\rho) = 1 \right. \right\},$$

an action with finite stabilizers.

From the correspondence between quotients and constellations explained in §2.2, it is clear that the open subscheme $\Omega^G(\mathcal{H}, h)$ defined in Lemma 2.2 is Γ -stable, and that there is a one-to-one correspondence between the Γ -orbits in $\Omega^G(\mathcal{H}, h)$ and the isomorphism classes of (G, h) -constellations. Therefore, we are naturally interested in performing the GIT quotient of $\mathrm{Quot}^G(\mathcal{H}, h)$ by Γ . However, to construct such a quotient we first need to fix a Γ -linearized ample line bundle on $\mathrm{Quot}^G(\mathcal{H}, h)$; the latter will depend on a finite subset $D \subset \mathrm{Irr} G$.

Proposition 2.4. ([BT15, §2.1]) *There exists a finite subset $D \subset \mathrm{Irr} G$ (depending on h and θ) and, for each $\rho \in D$, a finite dimensional vector space H_ρ such that there is a locally closed immersion*

$$(12) \quad \eta: \mathrm{Quot}^G(\mathcal{H}, h) \hookrightarrow \prod_{\rho \in D} \mathbb{P}(\Lambda^{h(\rho)} H_\rho).$$

Let us note that if $h(\rho) = 0$ for any $\rho \in D$, then $\mathbb{P}(\Lambda^{h(\rho)} H_\rho)$ is a point, and thus ρ plays no role in the embedding (12). Therefore, we can assume that $h(\rho) \neq 0$ for all $\rho \in D$. Also, as noticed in [BT15, Remark 2.2], for any set D' containing D we again obtain an embedding of $\mathrm{Quot}^G(\mathcal{H}, h)$. Hence, adding further representations if necessary, we will always assume that the following hypothesis holds.

Hypothesis 2.5. *D is a finite subset of $\mathrm{Irr} G$ such that the morphism (12) is a closed immersion, D contains D_- and intersects D_+ , and $h(\rho) \neq 0$ for every $\rho \in D$ (i.e., D is contained in $\mathrm{supp} h$).*

Choose a sequence of positive integers $(\kappa_\rho)_{\rho \in D} \in \mathbb{N}_{>0}^D$ and consider the ample line bundles $\mathcal{O}_\rho(\kappa_\rho)$ on $\mathbb{P}(\Lambda^{h(\rho)} H_\rho)$ which together give an ample line bundle

$$(13) \quad \mathcal{L} = \eta^* \left(\bigotimes_{\rho \in D} \mathcal{O}_\rho(\kappa_\rho) \right)$$

on $\mathrm{Quot}^G(\mathcal{H}, h)$. The action of Γ on $\mathrm{Quot}^G(\mathcal{H}, h)$ induces a natural linearization on some power \mathcal{L}^k of \mathcal{L} ; see the remark after [HL10, Lemma 4.3.2]. Replacing κ_ρ by $k\kappa_\rho$ for each $\rho \in D$, we can assume that \mathcal{L} itself carries a Γ -linearization.

Further, let $\chi: \Gamma \rightarrow \mathbb{C}^*$ be a character of Γ . Then $\chi(\gamma) = \prod_{\rho \in D_-} \det(\gamma_\rho)^{\chi_\rho}$ with $(\chi_\rho)_{\rho \in D_-} \in \mathbb{Z}^{D_-}$. We write \mathcal{L}_χ for the ample line bundle \mathcal{L} equipped with the linearization twisted by the character χ ; this ample line bundle depends on D

by construction. Finally, we denote by $\text{Quot}^G(\mathcal{H}, h)_D^{(s)s}$ the open subset of GIT-(semi)stable points of $\text{Quot}^G(\mathcal{H}, h)$ with respect to \mathcal{L}_χ ; see [MFK94, Definition 1.7] for the definition of GIT-(semi)stable points. One should really keep in mind that $\text{Quot}^G(\mathcal{H}, h)_D^{(s)s}$ does depend on D and that different choices of D lead to different sets of GIT-(semi)stable points.

Remark 2.6. In the following, we will consider \mathcal{L}_χ with $\kappa_\rho \in \mathbb{Q}_{>0}$ ($\rho \in D$) and $\chi_\rho \in \mathbb{Q}$ ($\rho \in D_-$). In that case, it has to be understood that we replace each κ_ρ by $p_1 \kappa_\rho$ and each χ_ρ by $p_2 \chi_\rho$, where p_1 resp. p_2 , is the least common multiple of the denominators of all the κ_ρ resp. of all the χ_ρ .

2.4. Choice of GIT parameters. In this subsection, we fix the values of the GIT parameters κ_ρ ($\rho \in D$) and χ_ρ ($\rho \in D_-$) in order to relate the GIT-(semi)stability for points of $\text{Quot}^G(\mathcal{H}, h)$ with the μ_θ -(semi)stability introduced for (G, h) -constellations in §1.1; see Theorem 2.7 for a precise statement. Let us mention that the numerical values given in [BT15, §3.3] are not correct and should be replaced by the numerical values given here.

Recall that we fixed a stability function θ at the beginning of §2, and let D be a finite subset of $\text{Irr } G$ satisfying Hypothesis 2.5. We denote:

$$\begin{aligned} A &:= \bigoplus_{\rho \in D_-} A_\rho, \text{ which is a vector space of dimension } r(h) = \sum_{\rho \in D_-} h(\rho); \\ d &:= \#(D \setminus D_-) \in \mathbb{N}_{>0}; \text{ and} \\ S_D &:= \sum_{\rho \in \text{Irr } G \setminus D} \theta_\rho h(\rho) \in \mathbb{Q}_{>0}. \end{aligned}$$

Given numbers κ_ρ ($\rho \in D$) and χ_ρ ($\rho \in D_-$), and any Hilbert function $h' : \text{Irr } G \rightarrow \mathbb{N}$, we denote

$$\begin{aligned} \kappa_D(h') &:= \sum_{\rho \in D} \kappa_\rho h'(\rho); \text{ and} \\ \chi(h') &:= \sum_{\rho \in D_-} \chi_\rho h'(\rho); \end{aligned}$$

where we stress the fact that the parameters κ_ρ ($\rho \in D$) depend on D . We now fix the following values for the κ_ρ ($\rho \in D$) and the χ_ρ ($\rho \in D_-$) introduced in §2.3:

$$(14) \quad \begin{cases} \kappa_\rho & \in \mathbb{Q}_{>0} & \text{for } \rho \in D_- \\ \kappa_\rho & = \theta_\rho + \frac{S_D}{d \cdot h(\rho)} & \text{for } \rho \in D \setminus D_- \\ \chi_\rho & = \theta_\rho - \kappa_\rho + \frac{\kappa(h)}{r(h)} & \text{for } \rho \in D_- \end{cases}$$

We recall that if $\rho \in D$, then $h(\rho) \neq 0$ by Hypothesis 2.5. Let us also note that

$$\kappa(h) := \kappa_D(h) = \sum_{\rho \in D_-} \kappa_\rho h(\rho) + \sum_{\rho \in D_+} \theta_\rho h(\rho)$$

is well-defined, i.e., it does not depend on D . Moreover, $S_D = -\sum_{\rho \in D} \theta_\rho h(\rho) \in \mathbb{Q}_{>0}$, so $\kappa_\rho \in \mathbb{Q}_{>0}$ for all $\rho \in D$.

Since (G, h) -constellations identify with elements of $\Omega^G(\mathcal{H}, h)$ (see §2.2), it makes sense to talk about μ_θ -(semi)stability for quotients. We define

$$\Omega^G(\mathcal{H}, h)_\theta^{(s)s} := \{ [q : \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \Omega^G(\mathcal{H}, h) \mid \mathcal{F} \text{ is } \mu_\theta\text{-(semi)stable} \}.$$

The latter is an open subscheme of $\Omega^G(\mathcal{H}, h)$ by Lemma 1.6 and [BT15, §4.1]. The next result motivates the choices we made for the values of κ_ρ and χ_ρ .

Theorem 2.7. *With the notation above and the GIT parameters given by (14), there exists a finite subset $D \subset \text{Irr } G$ big enough (i.e., D contains a given \tilde{D} and satisfies Hypothesis 2.5), such that*

$$\Omega^G(\mathcal{H}, h)_\theta^s \subseteq \text{Quot}^G(\mathcal{H}, h)_D^s \subseteq \text{Quot}^G(\mathcal{H}, h)_D^{ss} \subseteq \Omega^G(\mathcal{H}, h).$$

Moreover, each of these sets is Γ -stable and open in $\text{Quot}^G(\mathcal{H}, h)$.

Proof. The first inclusion is [BT15, Theorem 3.10], the third inclusion is [BT15, Lemma 3.1], and the last statement follows from the definition of GIT-(semi)stability and from [BT15, Proposition 4.1]. \square

Since the set of GIT-(semi)stable points of $\Omega^G(\mathcal{H}, h)$ is stable under the action of Γ , it makes sense to talk about GIT-(semi)stable (G, h) -constellations instead of GIT-(semi)stable quotients. Indeed, if \mathcal{F} is a (G, h) -constellation and there exist isomorphisms $\psi_\rho: A_\rho \rightarrow \mathcal{F}_\rho$ (see the beginning of §2.2) such that the corresponding quotient $[q: \mathcal{H} \twoheadrightarrow \mathcal{F}]$ is GIT-(semi)stable, then the same is true for any other choice of isomorphisms (by Γ -stability of the set of GIT-(semi)stable points).

Therefore, the GIT-stability can be seen as a stability condition on (G, h) -constellations. A set-theoretical version of Theorem 2.7 is given by

Corollary 2.8. *Let μ_θ be the stability condition of Definition 1.5. Then for any finite subset $D \subset \text{Irr } G$ big enough, we have the inclusions*

$$\left\{ \begin{array}{c} \mu_\theta\text{-stable} \\ (G, h)\text{-constellations} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{GIT-stable} \\ (G, h)\text{-constellations} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{GIT-semistable} \\ (G, h)\text{-constellations} \end{array} \right\}.$$

2.5. μ_D -stability and μ_D -Harder-Narasimhan filtration. In this subsection we introduce a new stability condition on the (\mathcal{O}_X, G) -modules generated in D_- , the μ_D -stability, which will be proved to coincide with the GIT-stability for (G, h) -constellations (Corollary 2.16). This reformulation of the GIT-stability in terms of the slope μ_D will ultimately allow us to construct another Harder-Narasimhan filtration for (G, h) -constellations (Theorem 2.17).

We fix a finite subset $D \subset \text{Irr } G$ satisfying Hypothesis 2.5 and we keep the notation introduced in §§2.1–2.4.

Lemma 2.9. ([BT15, §2.3]) *Let $[q: \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \text{Quot}^G(\mathcal{H}, h)$ and let $\lambda: \mathbb{C}^* \rightarrow \Gamma$ be a 1-parameter subgroup. Then*

$$[\bar{q}] := \lim_{t \rightarrow \infty} [q] \cdot \lambda(t)$$

is a well-defined element of $\text{Quot}^G(\mathcal{H}, h)$, which is a fixed point for the action of λ . In particular, λ acts linearly on the fibre $\mathcal{L}_\chi(\bar{q})$, where \mathcal{L}_χ is the Γ -linearized ample line bundle on $\text{Quot}^G(\mathcal{H}, h)$ defined at the end of §2.3.

Definition 2.10. Given $[q: \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \text{Quot}^G(\mathcal{H}, h)$ and a 1-parameter subgroup λ of Γ , we denote by $\mu_{\mathcal{L}_\chi}(q, \lambda)$ the weight for the action of λ on the fiber $\mathcal{L}_\chi([\bar{q}])$.

In our situation, the Hilbert-Mumford numerical criterion [MFK94, Theorem 2.1] can be formulated as follows:

Theorem 2.11. *The point $[q: \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \text{Quot}^G(\mathcal{H}, h)$ is GIT-(semi)stable if and only if $\mu_{\mathcal{L}_\chi}(q, \lambda) \gtrless 0$ for all non-trivial 1-parameter subgroups $\lambda: \mathbb{C}^* \rightarrow \Gamma$.*

After computing the weight $\mu_{\mathcal{L}_X}(q, \lambda)$ in terms of the GIT parameters of §2.4, we can rewrite the Hilbert-Mumford numerical criterion as follows:

Proposition 2.12. ([BT15, Proposition 2.11]) *The point $[q : \mathcal{H} \rightarrow \mathcal{F}] \in \text{Quot}^G(\mathcal{H}, h)$ is GIT-(semi)stable if and only if for all graded subspaces $0 \neq A' \subsetneq A$, that is $A' = \bigoplus_{\rho \in D_-} A'_\rho$ with $A'_\rho \subseteq A_\rho$ for every $\rho \in D_-$, the inequality*

$$(15) \quad \dim A \cdot (\kappa_D(\mathcal{F}') + \chi(A')) - \dim A' \cdot \kappa(h) \geq 0$$

holds, where $\mathcal{F}' := q(\bigoplus_{\rho \in D_-} A'_\rho \otimes V_\rho \otimes \mathcal{O}_X)$, and A , κ_D , and χ are defined in §2.4.

We follow the notation of §2.2. If $[q : \mathcal{H} \rightarrow \mathcal{F}] \in \Omega^G(\mathcal{H}, h)$, then we may establish a correspondence between subsheaves of \mathcal{F} generated in D_- and certain graded subspaces of A . Let $A' \subseteq A$ be a graded subspace, and let

$$(16) \quad \mathcal{F}' := q \left(\bigoplus_{\rho \in D_-} A'_\rho \otimes V_\rho \otimes \mathcal{O}_X \right) = \mathcal{O}_X \cdot \left(\sum_{\rho \in D_-} \varphi_\rho(A'_\rho)(V_\rho) \right)$$

be the (\mathcal{O}_X, G) -submodule of \mathcal{F} generated by the $\varphi_\rho(A'_\rho)$. Since $\varphi_\rho|_{A'_\rho}$ is injective, we have $\dim A'_\rho \leq \dim \mathcal{F}'_\rho$ for every $\rho \in D_-$. Now we define

$$\tilde{A}'_\rho := \varphi_\rho^{-1}(\mathcal{F}'_\rho) \quad \text{and} \quad \tilde{A}' := \bigoplus_{\rho \in D_-} \tilde{A}'_\rho.$$

Roughly speaking, \tilde{A}' is the biggest graded subspace of A which generates \mathcal{F}' . For this reason, we call $\tilde{A}' \subseteq A$ the *saturation of A'* .

Corollary 2.13. *The point $[q : \mathcal{H} \rightarrow \mathcal{F}] \in \Omega^G(\mathcal{H}, h)$ is GIT-(semi)stable if and only if inequality (15) holds for all saturated graded subspaces of A .*

Proof. The "only if" part is given by Proposition 2.12. For the "if" part, the proof is analogous to that of [BT15, Theorem 3.5]. Let $A' \subseteq A$ be a graded subspace, let \tilde{A}' be the saturation of A' , and let \mathcal{F}' be the subsheaf of \mathcal{F} generated by A' . Assuming that the inequality (15) holds for \tilde{A}' , we want to prove that it also holds for A' . If $\tilde{A}' = A'$, then we are done. Otherwise, $A' \subsetneq \tilde{A}'$ and we have

$$\begin{aligned} \chi(\tilde{A}') - \chi(A') &= \sum_{\rho \in D_-} \chi_\rho \cdot \dim(\tilde{A}'/A')_\rho \\ &< \sum_{\rho \in D_-} \frac{\kappa(h)}{\dim A} \cdot \dim(\tilde{A}'/A')_\rho, \quad \text{by definition of the } \chi_\rho, \\ &= \frac{\kappa(h) \cdot \dim(\tilde{A}'/A')}{\dim A} = \kappa(h) \cdot \frac{\dim \tilde{A}' - \dim A'}{\dim A}. \end{aligned}$$

It follows that

$$\dim A \cdot (\kappa_D(\mathcal{F}') + \chi(A')) - \dim A' \cdot \kappa(h) > \dim A \cdot (\kappa_D(\mathcal{F}') + \chi(\tilde{A}')) - \dim \tilde{A}' \cdot \kappa(h) \geq 0,$$

where the right inequality holds by assumption. Therefore, the inequality (15) holds for all graded subspaces $A' \subseteq A$. \square

Definition 2.14. Let $D \subset \text{Irr } G$ be a finite subset satisfying Hypothesis 2.5, and let \mathcal{F} be an (\mathcal{O}_X, G) -module generated in D_- . We say that \mathcal{F} is μ_D -(semi)stable if, for all (\mathcal{O}_X, G) -submodules $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$ generated in D_- , we have

$$\mu_D(\mathcal{F}') \leq \mu_D(\mathcal{F}),$$

where

$$\mu_D(\mathcal{F}') := \frac{-\kappa_D(\mathcal{F}') - \chi(\mathcal{F}')}{r(\mathcal{F}')} + \frac{\kappa(h)}{r(h)} \quad (\kappa_D \text{ and } \chi \text{ are defined in §2.4})$$

is the D -slope of \mathcal{F}' (and similarly for \mathcal{F}). If \mathcal{F} is not μ_D -semistable, we say that it is μ_D -unstable.

Remark 2.15. Note that the term $\frac{\kappa(h)}{r(h)}$ in the definition of μ_D is constant and does not affect the comparison between slopes in Definition 2.14. The reason to place it there is to simplify the comparison in §3 between μ_θ and μ_D -stability.

One easily checks that if \mathcal{F} is a (G, h) -constellation, then $\mu_D(\mathcal{F}) = 0$, independently of D . We saw at the end of §2.4 that we can talk about GIT-(semi)stable (G, h) -constellations; the next result makes the connection between GIT-stability and μ_D -stability.

Corollary 2.16. *Let \mathcal{F} be a (G, h) -constellation, and let μ_D be the stability condition of Definition 2.14. Then \mathcal{F} is GIT-(semi)stable if and only if \mathcal{F} is μ_D -(semi)stable.*

Proof. We have seen a little bit earlier that there is a correspondence between saturated graded subspaces $0 \subsetneq \widetilde{A}' \subsetneq A$ and (\mathcal{O}_X, G) -submodules $0 \subsetneq \mathcal{F}' \subsetneq \mathcal{F}$ generated in D_- . In particular, we have $\dim \widetilde{A}' = r(\mathcal{F}')$, $\chi(\widetilde{A}') = \chi(\mathcal{F}')$, and $\kappa_D(\widetilde{A}') = \kappa_D(\mathcal{F}')$. Then the result follows from Corollary 2.13. \square

We can finally state the main result of this subsection:

Theorem 2.17. *Let $D \subset \text{Irr } G$ be a finite subset satisfying Hypothesis 2.5, and let μ_D be as in Definition 2.14. Let \mathcal{F} be a (G, h) -constellation. Then \mathcal{F} has a unique filtration*

$$0 \subsetneq \mathcal{G}_1^D \subsetneq \mathcal{G}_2^D \subsetneq \cdots \subsetneq \mathcal{G}_{p_D}^D \subsetneq \mathcal{G}_{p_D+1}^D = \mathcal{F}$$

verifying

- (i) each \mathcal{G}_i^D is an (\mathcal{O}_X, G) -submodule generated in D_- ;
- (ii) each quotient $\mathcal{G}^{D,i} := \mathcal{G}_i^D / \mathcal{G}_{i-1}^D$ is μ_D -semistable; and
- (iii) the slopes of the quotients are strictly decreasing

$$\mu_D(\mathcal{G}^{D,1}) > \mu_D(\mathcal{G}^{D,2}) > \cdots > \mu_D(\mathcal{G}^{D,p_D}) > \mu_D(\mathcal{G}^{D,p_D+1}).$$

We call this filtration the μ_D -Harder-Narasimhan filtration (μ_D -HN filtration for short) of \mathcal{F} .

Proof. The proof is analogous to the one of Theorem 1.7. First we note that the functions κ_D , χ and r are additive on exact sequences (because so are the Hilbert functions). Hence, for any (\mathcal{O}_X, G) -submodule $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$ with a non-zero negative part, we can write

$$\mu_D(\mathcal{F}') = \frac{-\kappa_D(\mathcal{F}') - \chi(\mathcal{F}')}{r(\mathcal{F}')} + \frac{\kappa(h)}{r(h)} = \frac{-\kappa_D(\mathcal{F}') - \chi(\mathcal{F}') + r(\mathcal{F}') \frac{\kappa(h)}{r(h)}}{r(\mathcal{F}')}$$

as the quotient of two additive functions. Then it is clear that μ_D enjoys the *seesaw property* (see Lemma 1.8). Using the seesaw property and Proposition 1.9, we show the existence of a unique maximal μ_D -destabilizing subobject $\mathcal{F}' \subseteq \mathcal{F}$ (see Proposition 1.10). From this, we easily obtain the existence of the μ_D -HN filtration by induction on $r(\mathcal{F})$.

For the uniqueness part of the statement, Proposition 1.11 gives (after replacing the μ_θ -stability by the μ_D -stability) that necessarily $\mathcal{G}_1^D = \mathcal{F}'$, and we conclude by induction on the length of the filtration. \square

Let us note that the μ_D -HN filtration of a (G, h) -constellation \mathcal{F} is trivial if and only if \mathcal{F} is μ_D -semistable (equivalently, \mathcal{F} is GIT-semistable). Explicit examples of (G, h) -constellations with non trivial μ_D -HN filtration will be computed in §4.

3. COMPARISON BETWEEN THE DIFFERENT STABILITY NOTIONS

In this section we compare the θ or μ_θ -stability, introduced in §1.1, with the GIT or μ_D -stability, introduced in §2.5. In §3.1, we summarize the implications between these different stability notions for arbitrary (\mathcal{O}_X, G) -modules generated in D_- , and answer several questions remained open in [BT15]. Then, in §§3.2–3.3, we compare the μ_θ -HN and μ_D -HN filtrations. More precisely, in §3.2 we prove that when $D \subset \text{Irr } G$ is a finite subset big enough, the μ_θ -HN filtration is always a subfiltration of the μ_D -HN filtration (see Theorem 3.3). Next, in §3.3, we see how to attach to any (G, h) -constellation two convex polygons, the θ -polygon and the D -polygon, and we prove that the sequence of D -polygons converge to the θ -polygon when D grows (Theorem 3.7).

Throughout this section, θ is a stability function as in Definition 1.2, and $D \subset \text{Irr } G$ is a finite subset satisfying Hypothesis 2.5.

3.1. Relations between GIT-stability and θ -stability. We recall that the notions of θ -stability, μ_θ -stability, and μ_D -stability were defined for arbitrary (\mathcal{O}_X, G) -modules generated in D_- in §§1.1 and 2.5.

For all D *big enough* (i.e., $D \subset \text{Irr } G$ is a finite subset which contains a given \tilde{D} and satisfies Hypothesis 2.5), and for all (\mathcal{O}_X, G) -modules generated in D_- and contained in some (G, h) -constellation, we have the following implications:

$$(17) \quad \begin{array}{ccccccc} \theta\text{-stable} & \xrightarrow{(a)} & \mu_\theta\text{-stable} & \implies & \mu_\theta\text{-semistable} & \xleftarrow{(d)} & \theta\text{-semistable} \\ & \searrow & \Downarrow (b) & & \Uparrow (c) & & \\ & & \mu_D\text{-stable} & \implies & \mu_D\text{-semistable} & & \end{array}$$

where (a) and (d) follow from Lemma 1.6, (b) and (c) follow easily from the forthcoming Proposition 3.2, and the three other implications are straightforward. Also, for any (G, h) -constellation, (a) and (d) are equivalences by Lemma 1.6, and we have a notion of GIT-stability which coincide with the μ_D -stability by Corollary 2.16. In particular, for (G, h) -constellations, we have

$$\theta\text{-unstable} \iff \mu_\theta\text{-unstable} \implies \mu_D\text{-unstable} \iff \text{GIT-unstable}.$$

An important question, raised in [BT15], is the following:

Question. [BT15, Question 5.2] *Are the implications (b) and (c) in Diagram (17) equivalences for (G, h) -constellations?*

The answer is no in general. Indeed, there are examples of (G, h) -constellations which are μ_D -stable for all D big enough but never μ_θ -stable, and there are examples of (G, h) -constellations which are μ_θ -semistable but for which one can find D arbitrary big such that they are μ_D -unstable. We will compute explicitly such examples in §4.

Let us mention that, since the answer to [BT15, Question 5.2] is negative, it is clear that the answer to [BT15, Question 5.1], which is about the representability of a certain moduli functor, is also negative. However, we do not wish to pursue in this direction in this article.

Once we know that μ_θ -stability and μ_D -stability do not coincide, the next step is to "measure" the difference between these two stability notions. This will be performed in the next subsections.

3.2. Relations between the filtrations. In this section we prove one of the main results of this paper. We show that the μ_θ -HN filtration is always a subfiltration of the μ_D -HN filtration when $D \subset \text{Irr } G$ is a finite subset big enough (Theorem 3.3).

First, we need two preliminary results. The first one assures that (\mathcal{O}_X, G) -modules generated in D_- are both Noetherian and Artinian. The second one assures the convergence $\mu_D(\cdot) \rightarrow \mu_\theta(\cdot)$, when D tends to $\text{supp } h$, for subsheaves and quotients of (G, h) -constellations.

Lemma 3.1. *Let \mathcal{F} be an (\mathcal{O}_X, G) -module generated in D_- . Every increasing chain*

$$\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \mathcal{F}_3 \subsetneq \cdots \subsetneq \mathcal{F}$$

and every decreasing chain

$$\cdots \supsetneq \mathcal{F}_3 \supsetneq \mathcal{F}_2 \supsetneq \mathcal{F}_1 \supsetneq \mathcal{F}$$

of (\mathcal{O}_X, G) -submodules generated in D_- has length at most $r(\mathcal{F})$.

Proof. This result is a direct consequence of the definition of $r(\mathcal{F})$ in §1.1, using the fact that two (\mathcal{O}_X, G) -submodules $\mathcal{F}' \subseteq \mathcal{F}'' \subseteq \mathcal{F}$, both generated in D_- , coincide if and only if $r(\mathcal{F}') = r(\mathcal{F}'')$. \square

Proposition 3.2. *We fix $\epsilon > 0$. There exists a finite subset $D_\epsilon \subset \text{Irr } G$ satisfying Hypothesis 2.5 such that for all (G, h) -constellations \mathcal{F} and all (\mathcal{O}_X, G) -submodules $0 \subseteq \mathcal{F}' \subsetneq \mathcal{F}'' \subseteq \mathcal{F}$ generated in D_- , we have*

$$|\mu_\theta(\mathcal{F}''/\mathcal{F}') - \mu_D(\mathcal{F}''/\mathcal{F}')| < \epsilon$$

for all $D \supset D_\epsilon$.

Proof. We fix $D \subset \text{Irr } G$ a finite subset satisfying Hypothesis 2.5. Let h' be an arbitrary Hilbert function whose support intersects D_- and such that $h' \leq h$, where \leq denotes the partial order defined by (2). Note that all Hilbert functions corresponding to quotients $\mathcal{F}''/\mathcal{F}'$ of subsheaves $0 \subseteq \mathcal{F}' \subsetneq \mathcal{F}'' \subseteq \mathcal{F}$ are of this kind.

From Definition 2.14 and the choice of GIT parameters §2.4, a direct calculation gives

$$\mu_D(h') = \frac{-\sum_{\rho \in D} \theta_\rho h'(\rho) - \frac{S_D}{d} \sum_{\rho \in D \setminus D_-} \frac{h'(\rho)}{h(\rho)}}{r(h')}.$$

We deduced from this that the difference between μ_θ and μ_D is given by

$$\mu_\theta(h') - \mu_D(h') = \frac{-\sum_{\rho \notin D} \theta_\rho h'(\rho) + \frac{S_D}{d} \sum_{\rho \in D \setminus D_-} \frac{h'(\rho)}{h(\rho)}}{r(h')}.$$

Since $h' \leq h$, we have $\sum_{\rho \in D \setminus D_-} \frac{h'(\rho)}{h(\rho)} \leq d$, and

$$0 \leq \sum_{\rho \notin D} \theta_\rho h'(\rho) \leq \sum_{\rho \notin D} \theta_\rho h(\rho) = S_D.$$

Now, it is clear from the definition of S_D that for every $\epsilon > 0$, we can find D_ϵ satisfying Hypothesis 2.5 such that if $D \supset D_\epsilon$, then $S_D < \frac{\epsilon}{2}$. Thus,

$$|\mu_\theta(h') - \mu_D(h')| \leq \frac{2S_D}{r(h')} < \epsilon.$$

□

Let \mathcal{F} be a (G, h) -constellation, and let $0 \subseteq \mathcal{F}' \subsetneq \mathcal{F}'' \subseteq \mathcal{F}$ be (\mathcal{O}_X, G) -submodules generated in D_- . We denote by $\mathcal{A} = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$ the set of pair-wise different possible values for $\mu_\theta(\mathcal{F}''/\mathcal{F}')$. The fact that \mathcal{A} is finite follows from Proposition 1.9. Then, we define

$$(18) \quad \epsilon_0 := \frac{1}{4} \min_{i \neq j} |\alpha_i - \alpha_j|.$$

The next Theorem says that, given a finite subset $D \subset \text{Irr } G$ big enough, all terms of the μ_θ -HN filtration already appear in the μ_D -HN filtration, although the latter can contain more terms in general.

Theorem 3.3. *Let $D \subset \text{Irr } G$ be a finite subset satisfying Hypothesis 2.5 and containing D_{ϵ_0} , where ϵ_0 is defined by (18) and D_{ϵ_0} is given by Proposition 3.2. Let \mathcal{F} be a (G, h) -constellation. We denote the μ_θ -HN filtration of \mathcal{F} by*

$$0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \mathcal{F}_3 \subsetneq \dots \subsetneq \mathcal{F}_t \subsetneq \mathcal{F}_{t+1} = \mathcal{F},$$

and the μ_D -HN filtration of \mathcal{F} (where we drop the index D to simplify the notation) by

$$0 \subsetneq \mathcal{G}_1 \subsetneq \mathcal{G}_2 \subsetneq \mathcal{G}_3 \subsetneq \dots \subsetneq \mathcal{G}_p \subsetneq \mathcal{G}_{p+1} = \mathcal{F}.$$

Then, $p \geq t$ and there exists a subset of indices $1 \leq i_1 < i_2 < \dots < i_t < i_{t+1} = p+1$ such that $\mathcal{F}_j = \mathcal{G}_{i_j}$ for all $1 \leq j \leq t+1$. Moreover, for all $j = 1, \dots, t+1$ and all $k = i_{j-1} + 1, \dots, i_j$, we have $\mu_\theta(\mathcal{F}_j/\mathcal{F}_{j-1}) = \mu_\theta(\mathcal{G}_k/\mathcal{G}_{i_{j-1}})$.

Proof. The proof goes by induction on t . If $t = 0$, then the first part of the result is obvious, the second part follows from Proposition 3.2. We now suppose that $t \geq 1$. We want to prove that there exists $i_1 \geq 1$ such that $\mathcal{G}_{i_1} = \mathcal{F}_1$. If $\mathcal{G}_1 = \mathcal{F}_1$, then we are done. Otherwise let us prove that $\mathcal{G}_1 \subsetneq \mathcal{F}_1$ and that $\mathcal{F}_1/\mathcal{G}_1$ is a μ_D -destabilizing subobject of $\mathcal{F}/\mathcal{G}_1$. If $\mu_\theta(\mathcal{F}_1) > \mu_\theta(\mathcal{G}_1)$, then $\mu_D(\mathcal{F}_1) > \mu_D(\mathcal{G}_1)$ by definition of ϵ_0 and by Proposition 3.2, which contradicts the definition of \mathcal{G}_1 . Hence $\mu_\theta(\mathcal{F}_1) = \mu_\theta(\mathcal{G}_1)$, and thus $\mathcal{G}_1 \subsetneq \mathcal{F}_1$ by Proposition 1.11. The seesaw property (Lemma 1.8) gives $\mu_\theta(\mathcal{F}_1) = \mu_\theta(\mathcal{F}_1/\mathcal{G}_1)$ and $\mu_\theta(\mathcal{F}/\mathcal{G}_1) < \mu_\theta(\mathcal{F})$. We deduce that

$$\mu_\theta(\mathcal{F}/\mathcal{G}_1) < \mu_\theta(\mathcal{F}) < \mu_\theta(\mathcal{F}_1/\mathcal{G}_1)$$

and $\mathcal{F}_1/\mathcal{G}_1$ is a μ_θ -destabilizing subobject of $\mathcal{F}/\mathcal{G}_1$. Again, by using the definition of ϵ_0 and Proposition 3.2, we see that $\mathcal{F}_1/\mathcal{G}_1$ is a μ_D -destabilizing subobject of $\mathcal{F}/\mathcal{G}_1$.

Therefore $p \geq 2$ and $\mu_D(\mathcal{G}_2/\mathcal{G}_1) \geq \mu_D(\mathcal{F}_1/\mathcal{G}_1)$ by definition of \mathcal{G}_2 . If $\mathcal{G}_2 = \mathcal{F}_1$, then we are done. Otherwise, let us prove that $\mathcal{G}_2 \subsetneq \mathcal{F}_1$ and that $\mathcal{F}_1/\mathcal{G}_2$ is a μ_D -destabilizing subobject of $\mathcal{F}/\mathcal{G}_2$. We consider the exact sequence

$$0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G}_2 \rightarrow \mathcal{G}_2/\mathcal{G}_1 \rightarrow 0.$$

Since $\mu_\theta(\mathcal{G}_1) = \mu_\theta(\mathcal{F}_1)$ is maximal, the seesaw property gives

$$\mu_\theta(\mathcal{G}_1) \geq \mu_\theta(\mathcal{G}_2) \geq \mu_\theta(\mathcal{G}_2/\mathcal{G}_1).$$

If those inequalities were strict, using $\mu_\theta(\mathcal{G}_1) = \mu_\theta(\mathcal{F}_1/\mathcal{G}_1)$, the definition of ϵ_0 and Proposition 3.2, we would have

$$\mu_D(\mathcal{F}_1/\mathcal{G}_1) > \mu_D(\mathcal{G}_2) > \mu_D(\mathcal{G}_2/\mathcal{G}_1) \geq \mu_D(\mathcal{F}_1/\mathcal{G}_1),$$

where the last inequality is by definition of \mathcal{G}_2 , which is a contradiction. Hence, we necessarily have

$$\mu_\theta(\mathcal{F}_1) = \mu_\theta(\mathcal{G}_1) = \mu_\theta(\mathcal{G}_2) = \mu_\theta(\mathcal{G}_2/\mathcal{G}_1).$$

In particular, $\mathcal{G}_2 \subsetneq \mathcal{F}_1$ by Proposition 1.11. Also, the seesaw property implies that

$$\mu_\theta(\mathcal{F}/\mathcal{G}_2) < \mu_\theta(\mathcal{F}) < \mu_\theta(\mathcal{F}_1) = \mu_\theta(\mathcal{F}_1/\mathcal{G}_2),$$

and thus $\mathcal{F}_1/\mathcal{G}_2$ is a μ_θ -, therefore μ_D - by Proposition 3.2, destabilizing subobject of $\mathcal{F}/\mathcal{G}_2$.

Therefore $p \geq 3$ and $\mu_D(\mathcal{G}_3/\mathcal{G}_2) \geq \mu_D(\mathcal{F}_1/\mathcal{G}_2)$ by definition of \mathcal{G}_3 . If $\mathcal{G}_3 = \mathcal{F}_1$, then we are done. Otherwise we follow the same argument to construct subobjects $\mathcal{G}_4, \mathcal{G}_5$, etc, such that

$$0 \subsetneq \mathcal{G}_1 \subsetneq \mathcal{G}_2 \subsetneq \mathcal{G}_3 \subsetneq \mathcal{G}_4 \subsetneq \mathcal{G}_5 \subsetneq \cdots \subsetneq \mathcal{F}_1.$$

By Lemma 3.1, every increasing sequence has to stabilize, and thus there exists $1 \leq i_1 \leq p$ such that $\mathcal{G}_{i_1} = \mathcal{F}_1$. Moreover, it is clear that for all $1 \leq k \leq i_1$, we have $\mu_\theta(\mathcal{G}_k) = \mu_\theta(\mathcal{F}_1)$.

Now the μ_θ -HN filtration of $\mathcal{F}/\mathcal{F}_1$ is given by

$$0 \subsetneq \mathcal{F}_2/\mathcal{F}_1 \subsetneq \mathcal{F}_3/\mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_t/\mathcal{F}_1 \subsetneq \mathcal{F}_{t+1}/\mathcal{F}_1 = \mathcal{F}/\mathcal{F}_1,$$

and the μ_D -HN filtration of $\mathcal{F}/\mathcal{F}_1$ is given by

$$0 \subsetneq \mathcal{G}_{i_1+1}/\mathcal{F}_1 \subsetneq \mathcal{G}_{i_1+2}/\mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{G}_p/\mathcal{F}_1 \subsetneq \mathcal{G}_{p+1}/\mathcal{F}_1 = \mathcal{F}/\mathcal{F}_1.$$

Since the length of the μ_θ -HN filtration of $\mathcal{F}/\mathcal{F}_1$ is one less than the length of the μ_θ -HN filtration of \mathcal{F} , we conclude by induction. \square

We will see in §4 that μ_θ -HN and μ_D -HN filtrations need not coincide. Actually, the μ_D -HN filtration need not stabilize when D tends to supp h , even the number of terms of the μ_D -HN filtration can oscillate when D grows.

Remark 3.4. With the notation of Theorem 3.3, we saw that $\mu_\theta(\mathcal{F}_j/\mathcal{F}_{j-1}) = \mu_\theta(\mathcal{G}_k/\mathcal{G}_{i_{j-1}})$ for all $1 \leq j \leq t+1$ and all $i_{j-1}+1 \leq k \leq i_j$. It follows that

$$\mathcal{G}_{i_{j-1}+1}/\mathcal{G}_{i_{j-1}} \subsetneq \mathcal{G}_{i_{j-1}+2}/\mathcal{G}_{i_{j-1}} \subsetneq \cdots \subsetneq \mathcal{G}_{i_j}/\mathcal{G}_{i_{j-1}} = \mathcal{F}_j/\mathcal{F}_{j-1}$$

is a subfiltration of a (generally non-unique) μ_θ -Jordan-Hölder filtration (see Remark 1.12) of the μ_θ -semistable factor $\mathcal{F}_j/\mathcal{F}_{j-1}$.

3.3. Relations between the polygons. In this subsection, we explain how to associate a θ -polygon resp. a D -polygon, to any (G, h) -constellation \mathcal{F} . Those polygons usually appear in literature (see for instance [Sha77]) as a convenient way to encode numerical information regarding the μ_θ -HN and the μ_D -HN filtrations of \mathcal{F} . For instance, they encode the length of the filtration as well as the slope of each subsheaf. The "only" piece of information that we lose when considering those polygons instead of the actual filtrations is the explicit generators of each subsheaf.

Then we will prove that, even though the μ_θ -HN and μ_D -HN filtrations do not coincide in general, we have uniform convergence of the sequence of D -polygons to the θ -polygon when D grows (Theorem 3.7). Explicit examples of such polygons will be computed in §4.

Definition 3.5. Let \mathcal{F} be a (G, h) -constellation, and let \mathcal{F}_\bullet and \mathcal{G}_\bullet be the μ_θ -HN and μ_D -HN filtrations of \mathcal{F} respectively. We call θ -*polygon* of \mathcal{F} to the convex hull of the points with coordinates

$$(r(\mathcal{F}_i), w_i^\theta), \text{ where } w_i^\theta = r(\mathcal{F}_i) \cdot \mu_\theta(\mathcal{F}_i) .$$

Similarly, we call D -*polygon* to the convex hull of the points with coordinates

$$(r(\mathcal{G}_i), w_i^D), \text{ where } w_i^D = r(\mathcal{G}_i) \cdot \mu_D(\mathcal{G}_i) .$$

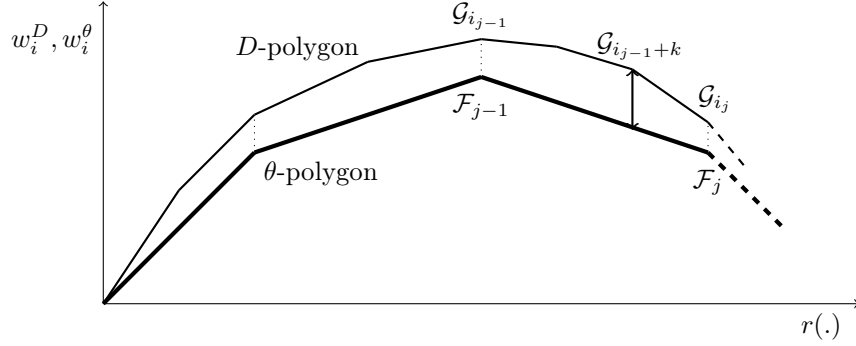


FIGURE 1. θ -polygon and D -polygon of a (G, h) -constellation

By construction, the θ -polygon and the D -polygons identify with the graph of concave piecewise linear functions

$$f_\theta, f_D : [0, r(h)] \rightarrow \mathbb{R}_{\geq 0}, \text{ where } f_\theta(r(\mathcal{F}_i)) = w_i^\theta \text{ and } f_D(r(\mathcal{G}_i)) = w_i^D ,$$

satisfying $f(0) = f(r(h)) = 0$. We can thus talk about convergence of a sequence of polygons.

Definition 3.6. Let f_θ resp. f_D , be the function whose graph is the θ -polygon resp. the D -polygon. Then we say that the sequence of D -polygons *converges uniformly* to the θ -polygon if

$$\forall \epsilon > 0, \exists \tilde{D} \subset \text{Irr } G, \forall D \supset \tilde{D} \text{ (satisfying Hypothesis 2.5), } \|f_D - f_\theta\|_\infty < \epsilon.$$

Theorem 3.7. *Let \mathcal{F} be a (G, h) -constellation. The sequence of D -polygons of \mathcal{F} converge uniformly to the θ -polygon of \mathcal{F} when the finite subset $D \subset \text{Irr } G$ (which satisfies Hypothesis 2.5, and thus is contained in $\text{supp } h$) converges to $\text{supp } h$.*

Proof. We fix $\epsilon > 0$, and we take D containing D_{ϵ_0} so that Theorem 3.3 holds. Let

$$0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \mathcal{F}_3 \subsetneq \cdots \subsetneq \mathcal{F}_t \subsetneq \mathcal{F}_{t+1} = \mathcal{F}$$

be the μ_θ -HN filtration of \mathcal{F} , and let

$$0 \subsetneq \mathcal{G}_1 \subsetneq \mathcal{G}_2 \subsetneq \cdots \subsetneq \mathcal{G}_{i_{j-1}} \subsetneq \cdots \subsetneq \mathcal{G}_{i_{j-1}+k} \subsetneq \cdots \subsetneq \mathcal{G}_{i_j} \subsetneq \cdots \subsetneq \mathcal{G}_p \subsetneq \mathcal{G}_{p+1} = \mathcal{F}$$

be the μ_D -HN filtration of \mathcal{F} , where $1 \leq i_1 < i_2 < \cdots < i_t < i_{t+1} = p+1$ are the indexes such that $\mathcal{F}_j = \mathcal{G}_{i_j}$ for all $1 \leq j \leq t+1$.

The functions f_θ and f_D , associated with the θ -polygon and the D -polygons respectively, are piecewise linear. Also, the set of abscissae of vertices of the θ -polygon is always contained in the set of abscissae of vertices of the D -polygon.

Consequently, to prove that the D -polygons converge to the θ -polygon, it suffices to bound

$$\max_{l=1,\dots,p} |f_D(r(\mathcal{G}_l)) - f_\theta(r(\mathcal{G}_l))|.$$

Let $\mathcal{G}_{i_{j-1}+k}$ be a term of the μ_D -HN filtration of \mathcal{F} . Then

$$\begin{aligned} & |f_D(r(\mathcal{G}_{i_{j-1}+k})) - f_\theta(r(\mathcal{G}_{i_{j-1}+k}))| \\ &= |w_{i_{j-1}+k}^D - (w_{j-1}^\theta + (r(\mathcal{G}_{i_{j-1}+k}) - r(\mathcal{G}_{i_{j-1}})) \mu_\theta(\mathcal{F}_j/\mathcal{F}_{j-1}))| \\ &= |r(\mathcal{G}_{i_{j-1}}) \mu_D(\mathcal{G}_{i_{j-1}}) + (r(\mathcal{G}_{i_{j-1}+k}) - r(\mathcal{G}_{i_{j-1}})) \mu_D(\mathcal{G}_{i_{j-1}+k}/\mathcal{G}_{i_{j-1}}) \\ &\quad - r(\mathcal{F}_{j-1}) \mu_\theta(\mathcal{F}_{j-1}) - (r(\mathcal{G}_{i_{j-1}+k}) - r(\mathcal{G}_{i_{j-1}})) \mu_\theta(\mathcal{F}_j/\mathcal{F}_{j-1})| \\ &\leq (r(\mathcal{G}_{i_{j-1}+k}) - r(\mathcal{G}_{i_{j-1}})) |\mu_D(\mathcal{G}_{i_{j-1}+k}/\mathcal{G}_{i_{j-1}}) - \mu_\theta(\mathcal{F}_j/\mathcal{F}_{j-1})| \\ &\quad + r(\mathcal{F}_{j-1}) |\mu_D(\mathcal{G}_{i_{j-1}}) - \mu_\theta(\mathcal{F}_{j-1})|. \end{aligned}$$

By Theorem 3.3, we have $\mu_\theta(\mathcal{G}_{i_{j-1}+k}/\mathcal{G}_{i_{j-1}}) = \mu_\theta(\mathcal{F}_j/\mathcal{F}_{j-1})$. We denote $\epsilon' := \frac{\epsilon}{r(h)}$, and we suppose that D contains $D_{\epsilon'}$. Then Proposition 3.2 implies that

$$|f_D(r(\mathcal{G}_{i_{j-1}+k})) - f_\theta(r(\mathcal{G}_{i_{j-1}+k}))| < (r(\mathcal{G}_{i_{j-1}+k}) - r(\mathcal{G}_{i_{j-1}}))\epsilon' + r(\mathcal{F}_{j-1})\epsilon' < \epsilon.$$

So denoting $\tilde{D} := D_{\epsilon_0} \cup D_{\epsilon'}$, we see that for all $D \supset \tilde{D}$, we have

$$\|f_D - f_\theta\|_\infty = \max_{l=1,\dots,p} |f_D(r(\mathcal{G}_l)) - f_\theta(r(\mathcal{G}_l))| < \epsilon.$$

□

4. EXAMPLES

In this last section we present several examples to illustrate the different phenomena that we considered throughout this article. By D we always mean a finite subset of $\text{Irr } G$ satisfying Hypothesis 2.5.

Since all these phenomena are already visible in small dimension, we will stick to the following quite simple framework. Let $G = \mathbb{G}_m$ be the multiplicative group. We recall that \mathbb{Z} identifies with $\text{Irr } G$, the set of isomorphism classes of irreducible representations of G , via the map $r \in \mathbb{Z} \mapsto V_r \in \text{Irr } G$, where V_r is the 1-dimensional representation on which $t \in G$ acts by multiplication by t^r . We consider the action of G on the algebra $\mathbb{C}[x, y]$ defined by $t.x := tx$ and $t.y := t^{-1}y$, for all $t \in G$. With this action, note that the weight of a monomial $x^a y^b$ is $a - b$. We take $X := \text{Spec } \mathbb{C}[x, y]/(xy)$. Then we have

$$\mathbb{C}[x, y]/(xy) \cong \mathbb{C}[x]_{>0} \oplus \mathbb{C} \oplus \mathbb{C}[y]_{>0} \cong \bigoplus_{r \in \mathbb{Z}} V_r$$

as G -modules. Let $h : \mathbb{Z} \rightarrow \mathbb{N}$ be the Hilbert function defined by $h(r) = 1$, for all $r \in \mathbb{Z}$. Then it is clear that \mathcal{O}_X is a (G, h) -constellation on X , provided that we choose θ such that D_- contains $\{0\}$ (to ensure that \mathcal{O}_X is generated in D_-).

We now consider the (\mathcal{O}_X, G) -submodules of \mathcal{O}_X . Those correspond to the G -stable ideals of \mathcal{O}_X and are of three kinds.

- (i) $I_p := (\overline{x}^p)$, with $p \geq 1$, then $h_{I_p}(r) = 1$ for $r \geq p$ and $h_{I_p}(r) = 0$ otherwise.
- (ii) $J_q := (\overline{y}^q)$, with $q \geq 1$, then $h_{J_q}(r) = 1$ for $r \leq -q$ and $h_{J_q}(r) = 0$ otherwise.
- (iii) $K_{p,q} := (\overline{x}^p, \overline{y}^q)$ with $p, q \geq 1$, then $h_{K_{p,q}}(r) = 1$ for $(r \geq p \text{ or } r \leq -q)$ and $h_{K_{p,q}}(r) = 0$ otherwise.

Here we denote by \bar{x} and \bar{y} the images of x and y in $\mathbb{C}[x, y]/(xy)$ respectively. Geometrically, X is simply the union of the two coordinate axes in the plane $\mathbb{A}_{\mathbb{C}}^2$, I_p is the ideal of the vertical thick line ($\bar{x}^p = 0$), J_q is the ideal of the horizontal thick line ($\bar{y}^q = 0$), and $K_{p,q}$ is the ideal of the thick point ($\bar{x}^p = 0 = \bar{y}^q$).

First, we begin with two examples to show that implications (b) and (c) of Diagram (17) are not equivalences in general. In particular, this answers negatively [BT15, Question 5.2]. To compute $\mu_D(h')$ in our forthcoming examples, we will use the formula

$$(19) \quad \mu_D(h') = \frac{-1}{r(h')} \left(\sum_{\rho \in D} \theta_{\rho} h'(\rho) + \frac{S_D}{d} \sum_{\rho \in D \setminus D_-} \frac{h'(\rho)}{h(\rho)} \right),$$

where S_D and d are defined in §2.4. This formula is obtained simply by plugging the numerical values given in §2.4 in Definition 2.14.

Example 4.1. *Example of a (G, h) -constellation μ_D -stable, for all finite subsets $D \subset \text{Irr } G$ big enough, but strictly μ_{θ} -semistable. Let θ be the stability function defined as follows:*

$$\begin{array}{cccccccccccccccccccc} r = & -k & \cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots & k \\ \theta_r = & \frac{1}{2^{k-2}} & \cdots & \frac{1}{4} & \frac{1}{2} & 0 & 0 & -1 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots & \frac{1}{2^{k-1}} \end{array}$$

We take $\mathcal{F} := \mathcal{O}_X$. Then we have $\theta(\mathcal{F}) = \sum_{r \in \mathbb{Z}} \theta_r h(r) = \sum_{r \in \mathbb{Z}} \theta_r = 0$, therefore θ satisfies the conditions of Definition 1.2. Since $D_- = \{0, 1\}$, the only (\mathcal{O}_X, G) -submodule of \mathcal{F} generated in D_- is $\mathcal{F}' := I_1$. We have

$$\theta(\mathcal{F}') = \sum_{r \in \mathbb{Z}} \theta_r h_{I_1}(r) = \sum_{r \geq 1} \theta_r = 0,$$

hence \mathcal{F} is strictly μ_{θ} -semistable, i.e. \mathcal{F} is μ_{θ} -semistable but not μ_{θ} -stable.

On the other hand, let us verify that \mathcal{F} is μ_D -stable when D is big enough. Let $D = D_N := [-N, N] \subset \mathbb{Z} = \text{Irr } G$. There exists $N_0 \geq 3$ such that D_N satisfies Hypothesis 2.5 for all $N \geq N_0$. An explicit computation with (19) gives

$$\begin{aligned} \mu_{D_N}(\mathcal{F}') &= \frac{-1}{r(\mathcal{F}')} \left(\sum_{r \in D_N} \theta_r h_{I_1}(r) + \frac{S_{D_N}}{d} \sum_{r \in D_N \setminus D_-} \frac{h_{I_1}(r)}{h(r)} \right) \\ &= (-1) \cdot \left(\frac{-1}{2^{N-1}} + \frac{\frac{3}{2^{N-1}}}{2N-1} (N-1) \right) = \frac{1}{2^{N-1}} \left(\frac{-N+2}{2N-1} \right) < 0. \end{aligned}$$

Hence, for all $N \geq N_0$, we have $\mu_{D_N}(\mathcal{F}') < \mu_{D_N}(\mathcal{F}) = 0$, i.e., \mathcal{F} is μ_{D_N} -stable. Finally, it is easy to check that the same holds for all D big enough.

Example 4.2. *Example of a (G, h) -constellation \mathcal{F} which is μ_{θ} -semistable but μ_D -unstable for some finite subsets $D \subset \text{Irr } G$ arbitrarily big. Let θ be the stability function defined as follows:*

$$\begin{array}{cccccccccccccccccccc} r = & -k & \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots & k \\ \theta_r = & 0 & \cdots & 0 & 0 & 1 & -1 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2^{k-1}} \end{array}$$

As in the previous example, we take $\mathcal{F} := \mathcal{O}_X$. Then we have $\theta(\mathcal{F}) = 0$, therefore θ satisfies the conditions of Definition 1.2. The only (\mathcal{O}_X, G) -submodule of \mathcal{F} generated in $D_- = \{0, 1\}$ is $\mathcal{F}' := I_1$, and we have $\theta(\mathcal{F}') = 0$, i.e., \mathcal{F} is strictly μ_θ -semistable.

On the other hand, let us verify that \mathcal{F} is μ_D -unstable for some finite subset $D \subset \text{Irr } G$ big enough. We denote again $D = D_N := [-N, N] \subset \mathbb{Z}$. There exists $N_0 \geq 1$ such that D_N satisfies Hypothesis 2.5 for all $N \geq N_0$. A similar computation using (19) gives

$$\begin{aligned} \mu_{D_N}(\mathcal{F}') &= \frac{-1}{r(\mathcal{F}')} \left(\sum_{r \in D_N} \theta_r h_{I_1}(r) + \frac{S_{D_N}}{d} \sum_{r \in D_N \setminus D_-} \frac{h_{I_1}(r)}{h(r)} \right) \\ &= (-1) \cdot \left(\frac{-1}{2^{N-1}} + \frac{1}{2^{N-1}} (N-1) \right) = \frac{1}{2^{N-1}} \left(\frac{N}{2N-1} \right) > 0. \end{aligned}$$

Hence, for all $N \geq N_0$, we have $\mu_{D_N}(\mathcal{F}') > \mu_{D_N}(\mathcal{F}) = 0$, i.e., \mathcal{F} is μ_{D_N} -unstable.

To summarize, in this example the μ_θ -HN filtration of \mathcal{F} is trivial (since \mathcal{F} is μ_θ -semistable), but the μ_{D_N} -HN filtration of \mathcal{F} is $0 \subsetneq \mathcal{F}' \subsetneq \mathcal{F}$ for all $N \geq N_0$. The latter is also the μ_θ -Jordan-Hölder filtration of \mathcal{F} ; see Remark 3.4. Figure 2 illustrates the behavior of the θ -polygon and the D_N -polygons for different N .

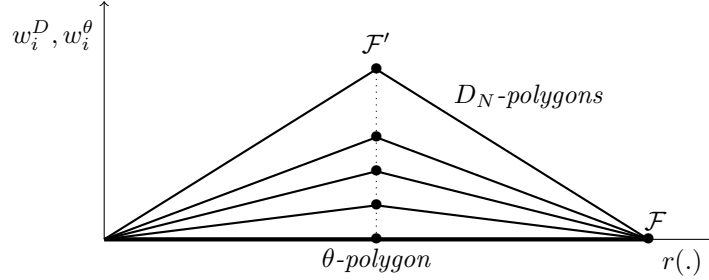


FIGURE 2. θ -polygon and D_N -polygons of Example 4.2

One might ask whether the reason for the μ_θ -HN and the μ_D -HN filtrations to be distinct in Example 4.2 is that the (G, h) -constellation \mathcal{F} is μ_θ -semistable (which "forces" the μ_θ -HN filtration to be trivial). The answer is actually negative as the next two examples will show.

Example 4.3. Example where the μ_θ -HN and μ_D -HN filtrations are both non-trivial, and the number of terms of the μ_D -HN filtration does not stabilize when the finite subset $D \subset \text{Irr } G$ grows. Let θ be the stability function defined as follows:

$$\begin{array}{cccccccccccccccccccccccc} r = & -k & \cdots & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & k \\ \theta_r = & \frac{1}{2^{\frac{k-2}{2}}} & \cdots & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 1 & 1 & -1 & -1 & -2 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \cdots & \frac{1}{2^{\frac{k-1}{2}}} \end{array}$$

Then, again, θ satisfies the conditions of Definition 1.2. We now have $D_- = \{0, 1, 2\}$, hence there are two (\mathcal{O}_X, G) -submodules of \mathcal{F} generated in D_- which are

$\mathcal{F}_1 := I_2$ and $\mathcal{F}_2 := I_1$, with $r(\mathcal{F}_1) = 1$ and $r(\mathcal{F}_2) = 2$. They verify

$$\mu_\theta(\mathcal{F}_1) = \mu_\theta(\mathcal{F}_2) = 1 > \mu_\theta(\mathcal{F}) = 0,$$

and thus the μ_θ -HN filtration of \mathcal{F} is

$$0 \subsetneq \mathcal{F}_2 \subsetneq \mathcal{F}.$$

Now, let us distinguish between two situations: the even and the odd cases. For the even case we take $D = D_{N,\text{even}} := [-2N - 2, 2N + 2]$, and for the odd case we take $D = D_{N,\text{odd}} := [-2N - 3, 2N + 3]$, where these bounds are chosen in order to simplify the calculations. In both cases, there exists $N_0 \geq 1$ such that $D_{N,\bullet}$ satisfies Hypothesis 2.5 for all $N \geq N_0$.

In the even case, we have

$$\begin{aligned} \mu_{D_{N,\text{even}}}(\mathcal{F}_1) &= \frac{-1}{r(\mathcal{F}_1)} \left(\sum_{r \in D_{N,\text{even}}} \theta_r h_{I_2}(r) + \frac{S_{D_{N,\text{even}}}}{d} \sum_{r \in D_{N,\text{even}} \setminus D_-} \frac{h_{I_2}(r)}{h(r)} \right) \\ &= (-1) \cdot \left(\left(-1 - \frac{1}{2^N}\right) + \frac{\frac{1}{2^{N-1}}}{4N+2} (2N) \right) = 1 + \frac{1}{2^N} \left(\frac{2}{4N+2} \right) ; \text{ and} \\ \mu_{D_{N,\text{even}}}(\mathcal{F}_2) &= \frac{-1}{r(\mathcal{F}_2)} \left(\sum_{r \in D_{N,\text{even}}} \theta_r h_{I_1}(r) + \frac{S_{D_{N,\text{even}}}}{d} \sum_{r \in D_{N,\text{even}} \setminus D_-} \frac{h_{I_1}(r)}{h(r)} \right) \\ &= \left(\frac{-1}{2} \right) \cdot \left(\left(-2 - \frac{1}{2^N}\right) + \frac{\frac{1}{2^{N-1}}}{4N+2} (2N) \right) = 1 + \frac{1}{2^{N+1}} \left(\frac{2}{4N+2} \right) . \end{aligned}$$

Observe that $\mu_{D_{N,\text{even}}}(\mathcal{F}_1) > \mu_{D_{N,\text{even}}}(\mathcal{F}_2) > 0$, hence the $\mu_{D_{N,\text{even}}}$ -HN filtration of \mathcal{F} is

$$0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \mathcal{F} .$$

Performing analogous calculations in the odd case, we get

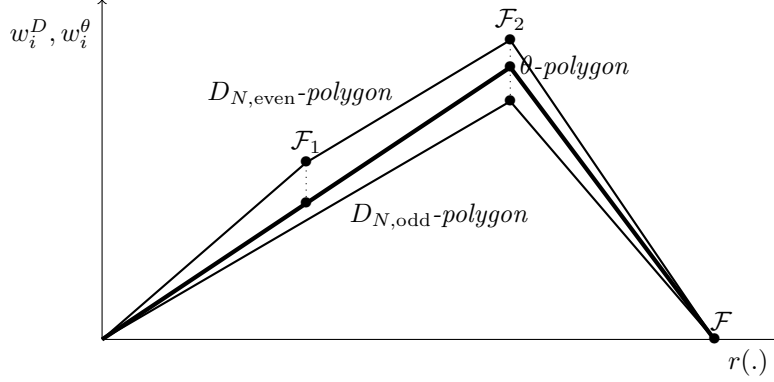
$$\begin{aligned} \mu_{D_{N,\text{odd}}}(\mathcal{F}_1) &= 1 + \frac{1}{2^{N+1}} \left(\frac{-2N+1}{4N+4} \right) ; \text{ and} \\ \mu_{D_{N,\text{odd}}}(\mathcal{F}_2) &= 1 + \frac{1}{2^{N+2}} \left(\frac{-2N+1}{4N+4} \right) . \end{aligned}$$

We see that $\mu_{D_{N,\text{odd}}}(\mathcal{F}_2) > \mu_{D_{N,\text{odd}}}(\mathcal{F}_1) > 0$, hence the $\mu_{D_{N,\text{odd}}}$ -HN filtration of \mathcal{F} is

$$0 \subsetneq \mathcal{F}_2 \subsetneq \mathcal{F} .$$

Therefore, in the odd case, the $\mu_{D_{N,\text{odd}}}$ -HN filtration has exactly the same terms as the μ_θ -HN filtration, but in the even case, the $\mu_{D_{N,\text{even}}}$ -HN filtration has one more term. Figure 3 illustrates this behavior by showing the θ -polygon and the D_N -polygons of \mathcal{F} . Also, observe how the $D_{N,\text{even}}$ -polygons lie above the θ -polygon, the $D_{N,\text{odd}}$ -polygons lie below, and both sequences of polygons converge to the θ -polygon when N grows (as stated by Theorem 3.7).

To compute our final example we will slightly change the framework and pick another Hilbert function h . Consider $G = \mathbb{G}_m$ with the same action on the algebra

FIGURE 3. θ -polygon and D_N -polygons of Example 4.3

$\mathbb{C}[x, y]$ as before, but take now $X := \text{Spec } \mathbb{C}[x, y]/(xy^2, x^3y)$. Then we have

$$\begin{aligned} \mathbb{C}[x, y]/(xy^2, x^3y) &\cong \mathbb{C}[x]_{>0} \oplus \mathbb{C} \oplus \mathbb{C} < \overline{xy} > \oplus \mathbb{C} < \overline{x^2y} > \oplus \mathbb{C}[y]_{>0} \\ &\cong \left(\bigoplus_{r \in \mathbb{Z} \setminus \{0,1\}} V_r \right) \oplus V_0^{\oplus 2} \oplus V_1^{\oplus 2}. \end{aligned}$$

as G -modules, where \overline{x} and \overline{y} denote the images of x and y in $\mathbb{C}[x, y]/(xy^2, x^3y)$. Let $h : \mathbb{Z} \rightarrow \mathbb{N}$ be the Hilbert function defined by $h(r) = 2$ if $r = 0, 1$, and $h(r) = 1$ for all $r \neq 0, 1$. It is clear that \mathcal{O}_X is a (G, h) -constellation on X , provided that we choose θ such that $\theta_0 < 0$ (to guarantee that \mathcal{O}_X is generated in D_-).

Example 4.4. Example where, for all D big enough, the μ_θ -HN and μ_D -HN filtrations are both non-trivial, and the μ_θ -HN filtration is a strict subfiltration of the μ_D -HN filtration which is, in turn, a strict subfiltration of some μ_θ -Jordan-Hölder filtration (see Remark 3.4). Let θ be defined as follows:

$$\begin{array}{cccccccccccccccccccc} r = & -k & \dots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \dots & k \\ \theta_r = & 0 & \dots & 0 & 0 & \frac{5}{8} & -1 & -1 & -2 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \dots & \frac{1}{2^{k-2}} \end{array}$$

Let $\mathcal{F} := \mathcal{O}_X$ and observe that $\theta(\mathcal{F}) = 0$, so that θ satisfies the conditions of Definition 1.2. There are five non-zero proper (\mathcal{O}_X, G) -submodules of \mathcal{F} generated in $D_- = \{0, 1, 2\}$, say

$$\mathcal{F}_1 := (\overline{x^2y}), \mathcal{F}_2 := (\overline{x^2}), \mathcal{F}_3 := (\overline{x^2}, \overline{xy}), \mathcal{F}_4 := (\overline{x}), \text{ and } \mathcal{F}_5 := (\overline{xy}).$$

These submodules verify

$$r(\mathcal{F}_1) = 1, r(\mathcal{F}_2) = 2, r(\mathcal{F}_3) = 3, r(\mathcal{F}_4) = 4, \text{ and } r(\mathcal{F}_5) = 2.$$

For the sake of completeness, we detail their Hilbert functions which are

$$h_{\mathcal{F}_1}(r) = \begin{cases} 1, & r = 1 \\ 0, & r \neq 1 \end{cases}; \quad h_{\mathcal{F}_2}(r) = \begin{cases} 1, & r \geq 1 \\ 0, & r \leq 0 \end{cases}; \quad h_{\mathcal{F}_3}(r) = \begin{cases} 1, & r \geq 0 \\ 0, & r \leq -1 \end{cases};$$

$$h_{\mathcal{F}_4}(r) = \begin{cases} 1, & r = 0 \\ 2, & r = 1 \\ 1, & r \geq 2 \\ 0, & r \leq -1 \end{cases}; \quad h_{\mathcal{F}_5}(r) = \begin{cases} 1, & r = 0, 1 \\ 0, & r \neq 0, 1 \end{cases}.$$

First, we see that

$$\mu_\theta(\mathcal{F}_1) = \mu_\theta(\mathcal{F}_2) = \mu_\theta(\mathcal{F}_3) = \mu_\theta(\mathcal{F}_4) = \mu_\theta(\mathcal{F}_5) = 1 > 0 = \mu_\theta(\mathcal{F}),$$

hence the μ_θ -HN filtration of \mathcal{F} is

$$0 \subsetneq \mathcal{F}_4 \subsetneq \mathcal{F}.$$

Now, let $D = D_N := [-N, N] \subset \mathbb{Z}$. There exists $N_0 \geq 1$ such that D_N satisfies Hypothesis 2.5 for all $N \geq N_0$. Let us determine the μ_{D_N} -HN filtration of \mathcal{F} . Analogous calculations to those performed in the previous examples give

$$\mu_{D_N}(\mathcal{F}_1) = 1;$$

$$\mu_{D_N}(\mathcal{F}_2) = 1 + \frac{1}{2^{N-1}} \left(\frac{N - \frac{3}{2}}{2N - 2} \right);$$

$$\mu_{D_N}(\mathcal{F}_3) = 1 + \frac{1}{3 \cdot 2^{N-2}} \left(\frac{N - 2}{2N - 2} \right);$$

$$\mu_{D_N}(\mathcal{F}_4) = 1 + \frac{1}{2^N} \left(\frac{N - \frac{5}{2}}{2N - 2} \right);$$

$$\mu_{D_N}(\mathcal{F}_5) = 1.$$

Also, asymptotically we have

$$\mu_{D_N}(\mathcal{F}_2) \sim 1 + \frac{1}{2^N};$$

$$\mu_{D_N}(\mathcal{F}_3) \sim 1 + \frac{1}{3 \cdot 2^{N-1}};$$

$$\mu_{D_N}(\mathcal{F}_4) \sim 1 + \frac{1}{2^{N+1}}.$$

Hence, there exists $N_1 \geq N_0$ such that, for all $N \geq N_1$, the μ_{D_N} -HN filtration of \mathcal{F} is

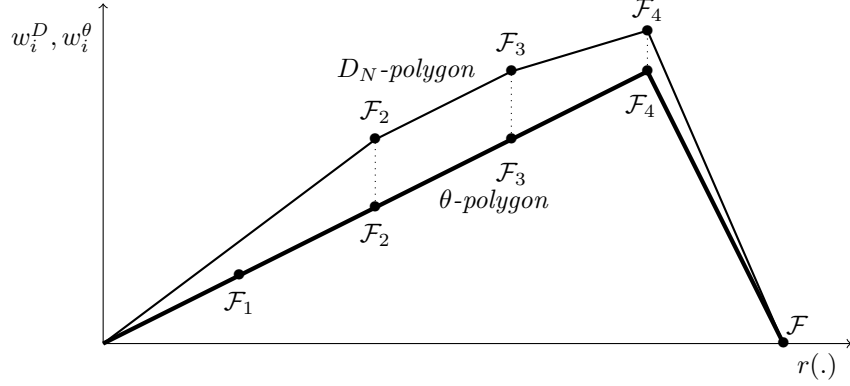
$$0 \subsetneq \mathcal{F}_2 \subsetneq \mathcal{F}_3 \subsetneq \mathcal{F}_4 \subsetneq \mathcal{F}.$$

Observe that, for $N \geq N_1$, the μ_{D_N} -HN filtration contains the unique non trivial term of the μ_θ -HN filtration, \mathcal{F}_4 , (as proved in Theorem 3.3) but it contains also two more terms, \mathcal{F}_2 and \mathcal{F}_3 . On the other hand, the first μ_θ -semistable factor of the μ_θ -HN filtration of \mathcal{F} , which is \mathcal{F}_4 , has two different μ_θ -Jordan-Hölder filtrations

$$0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \mathcal{F}_3 \subsetneq \mathcal{F}_4, \text{ and } 0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_5 \subsetneq \mathcal{F}_3 \subsetneq \mathcal{F}_4,$$

and the μ_{D_N} -HN filtration is a subfiltration of the first one but not of the second one (see Remark 3.4). Figure 4 illustrates the situation.

Remark 4.5. For the reader willing to consider fancier situations, other examples of (G, h) -constellations with G a classical group can be found in [Ter12].

FIGURE 4. θ -polygon and D_N -polygons of Example 4.4

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